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ASYMPTOTIC METHODS IN THE THEORY OF LINEAR DIFFERENTIAL EQUATIONS

by S. F. Feshchenko, N. I. Shkil', and L. D. Nikolenko

"Naukova Dumka" Publishing House, Kiev, 1966

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ASYMPTOTIC METHODS IN THE THEORY OF LINEAR DIFFERENTIAL EQUATIONS

By S. F. Feshchenko, N. I. Shkil', and L. D. Nikolenko

Translation of "Asimptoticheskiye metody v teorii lineynykh differentsial'nykh uravneniy."
Publishing House "Naukova Dumka" Kiev, 1966

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ANNOTATION

The book describes the asymptotic methods of integrating linear differential equations with slowly varying coefficients, which are encountered in many branches of physics and engineering.

The book is designed for a wide circle of engineers and scientists who are interested in the problems associated with the approximate integration of differential equations with variable coefficients and in particular with equations which describe oscillatory processes.

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INTRODUCTION

As we know, the solution of many problems in physics and engineering is reduced to the investigation of differential equations with variable coefficients. Since the exact solution of such an equation is possible only in exceptional cases, it becomes necessary to apply various approximate methods of integration. Among the approximate methods of integrating differential equations, an important position is occupied by the asymptotic methods which are based on the idea of expanding the unknown solution into a formal power series of some small parameter. Although in this case the power series, as a rule, diverge, the approximate solution which is obtained by cutting off the formal series at some m-th term, nevertheless, proves to be quite suitable for a whole series of practical calculations. The approximate solution obtained in this manner has an asymptotic nature in the sense that it approaches the corresponding exact solution not as m is increased, but when m is fixed and when the small parameter tends to zero. /3*

Since the asymptotic methods make it possible to obtain an analytical expression for the approximate solution, the latter is also suitable for investigating the qualitative picture of solution behavior over a rather large, although finite, interval of independent variable variation (see for example ref. 118).

The present monograph presents the asymptotic methods of integrating a definite class of linear differential equations, specifically those in which the coefficients are functions of slowly varying time $\tau = \epsilon t$, where ϵ is a small positive parameter which indicates that the coefficients of the equation vary slowly, i.e., their derivatives with respect to the independent variable t are proportional to the small parameter ϵ.

Differential equations with slowly varying coefficients are frequently encountered in practice. These include, for example, equations with a small parameter in front of higher derivatives, as have been investigated in the works of A. N. Tikhonov (refs. 78-80), I. S. Gradshteyn (refs. 16-17), V. M. Volosov (refs. 13-14), A. B. Vasil'yeva (refs. 9-11), K. V. Zadiraka (refs. 26-28), M. I. Vishik, L. A. Lyusternik et al. (ref. 12).

Indeed, let us consider the equation

$$\varepsilon \frac{dy}{dx} + p(x)y = 0. \tag{1}$$

If we let

$$x = \varepsilon t = \tau, \tag{2}$$ /4

we obtain

$$\frac{dy}{dt} + p(\tau)y = 0, \tag{3}$$

* Numbers in the margin indicate pagination in original foreign text.

i.e., we obtain an equation in which the coefficient $p(\tau)$ is a slowly varying function of the variable t.

Some problems with eigenvalues are also reduced to similar equations. For example, if in the Sturm-Liouville equation

$$\frac{d^2y}{dx^2} + [\lambda g(x) - r(x)]\, y = 0, \tag{4}$$

we have the equation

$$x = \varepsilon t \equiv \tau, \qquad \varepsilon = \frac{1}{\sqrt{\lambda}}, \tag{5}$$

where λ is a large parameter, we obtain the following equation

$$\frac{d^2y}{dt^2} + [g(\tau) - \varepsilon^2 r(\tau)]\, y = 0, \tag{6}$$

in which $g(\tau)$, $r(\tau)$ are slowly varying functions.

In addition to this (and this is particularly important in application), we encounter a whole series of practical problems (see for example Sections 5 and 10 of the present monograph) where it is possible to isolate a dimensionless combination of known quantities which play the role of the small parameter. This in turn makes it possible for us to examine the resulting equations as equations with slowly varying coefficients (see also refs. 64 and 103).

We shall now present a brief history of the problem.

The idea of representing the solutions of differential equations in asymptotic form had its origin in the works of Liouville (refs. 123 and 124). The fact is that the investigation of the convergence of an expanded assigned "arbitrary" function in terms of the eigenfunctions of some boundary value problem is based on the study of the behavior of the eigenfunctions themselves. Therefore, it is of substantial interest to obtain the asymptotic equations of the eigenfunctions. Liouville first obtained asymptotic equations for the solutions of second-order equations of type (4) and then for equations of higher orders.

In view of the importance attributed to the problem of the asymptotic representation of solutions to differential equations encountered in mathematical physics, this theory was rapidly developed after Liouville.

It was later established that the asymptotic representation of solutions is significant not only for the analysis of the convergence of expansions in terms of eigenfunctions, but also for many problems of practical value which had an entirely different nature. Thus, Fowler and Lock (ref. 132) applied the results of Liouville to the solution of the approximate equations of shell trajectory. De Sparre (ref. 127) used this method to solve the problem of a spinning shell. The asymptotic nature of the solution obtained by de Sparre was demonstrated by Horn (ref. 121). /5

The work of H. Poincaré (ref. 125) served as a substantial landmark in the development of the asymptotic representation of solutions. In this work Poincaré organized and substantially developed the ideas on this question.

The asymptotic method was applied with success and skill by the Russian scientist V. A. Steklov, particularly in his work "The Problem of the Cooling of a Heterogeneous Solid Body" (Khar'kov, 1896). However, all the investigations mentioned above revolve, so to speak, around the ideas of Liouville and pertain to the investigation of self-conjugate differential systems. We must therefore recognize that in spite of the general results which were obtained by investigations in this direction, the scope of application of this method remained limited.

We do not encounter these limitations in the works of Birkhoff (refs. 119 and 120), which generalize the results of Horn and Schlesinger to equations of the n-th order and to a system of equations.

We note that the asymptotic representation of solutions to differential equations which contain a parameter is determined generally speaking by the behavior of the roots of some algebraic equation which is similar to the characteristic equation for linear differential equations with constant coefficients. In the future this equation will also be called the characteristic equation.

The cited works of Schlesinger and Birkhoff considered the case when the roots of the characteristic equations remained simple along the entire interval over which the argument varied.

A further generalization of these works is contained in the work of Ya. D. Tamarkin. In his work (ref. 76) he considered partially the case of the multiple roots of a characteristic equation by constructing the asymptotic solution of a second-order linear system.

However, the Schlesinger-Birkhoff-Tamarkin theory pertained to ordinary homogeneous linear differential equations. It was necessary to generalize the theory of the asymptotic representation of solutions to nonhomogeneous differential equations. This gap was filled by Fowler and Lock (refs. 132 and 133), who also simplified substantially the proof of the basic results obtained by Birkhoff.

The work of W. Y. Tryizinski (ref. 131), which appeared in 1936, gave a complete state-of-the-art survey concerning the question of the asymptotic representation of solutions to systems of ordinary linear differential equations. Additionally, this work generalized the Schlesinger-Birkhoff-Tamarkin theory to the case of linear integro-differential equations with a parameter. /6

The following expression was usually used to give the asymptotic representation of solutions to differential equations containing a parameter:

$$e^{\alpha^r \sum_{s=0}^{r-1} \Omega_s(x)\alpha^{-s}} \left[\sum_{s=0}^{m} y_s(x)\alpha^{-s} + \eta\alpha^m \right],$$

where α is a large parameter, while η is a quantity which approaches zero when $\alpha \to \infty$ along a path which lies entirely in some region of the plane α.

A series of works by V. S. Pugachev which appeared from 1940 to 1946 (refs. 56-60) presented interesting and rather novel results relative to the theory of the asymptotic representation of integrals for nonhomogeneous ordinary differential equations of the second and higher orders whose coefficients contained a parameter. References 56 and 57 contain a substantially new result which pertains to the evaluation of errors incurred in the approximate representation of integrals by the first terms of their asymptotic expansions.

In addition to theorems which generalize the results of preceding investigators, Pugachev demonstrated a new type of asymptotic representation of integrals for systems of ordinary linear homogeneous and nonhomogeneous differential equations.

This new representation of a solution has the form

$$\sum_{h=1}^{n} z_h(x, \alpha)\left\{\sum_{k=0}^{m} y_{hk}(x)\alpha^{-k} + \eta_h \alpha^{-m}\right\},$$

where $z_h(x, \alpha)(h=1, 2, \ldots, n)$ are functions which satisfy a certain system of linear differential equations, while η_h are quantities which tend to 0 when $\alpha \to \infty$.

Pugachev also generalized the theory of asymptotic representation to the case of an arbitrary complex domain for the variation of arguments.

The works of Turrittin (refs. 130, 77) and Sibuya (ref. 126) also pertain to asymptotic methods. Basically these works accomplish an asymptotic partitioning of an initial system of linear differential equations into several subsystems of a lower order whose number is determined by the number of identically multiple roots of the characteristic equation.

Our short historic survey would be incomplete if we did not consider the asymptotic methods developed by N. M. Krylov and N. N. Bogolyubov (refs. 41, 2-4) for the solution of problems in nonlinear mechanics. These methods, which /7 were strictly substantiated, were used to describe both periodic and quasi-periodic processes and also to investigate the most general nonconservative systems.

It should be pointed out that the usual methods of expansion in powers of the small parameter lead to approximate solutions which contain the so-called secular terms in which the independent variable t is contained outside the sign of the trigonometric expressions. As a result of this, the error produced by the substitution of such approximate solutions into the differential equations decreases together with the small parameter when t is fixed, but this decrease is not uniform with respect to t. It is possible to find a sequence of values $t \to \infty$ for which the error will not tend to zero, but rather to infinity regardless of the rate at which the small parameter decreases. Therefore, the range

over which the conventional method of expansion in powers of small parameter is applicable must be limited by a sufficiently narrow interval of time. However, when oscillatory processes, particularly high-frequency processes, are investigated it is necessary to have approximate equations which are suitable for a more extended interval of time.

The asymptotic methods developed by N. M. Krylov and N. N. Bogolyubov lead to approximate equations which no longer contain the secular terms. The asymptotic solutions obtained in this manner are valid for a sufficiently large (although finite) interval of time.

A further significant development of the methods of nonlinear mechanics, proposed and justified by Krylov and Bogolyubov, is to be found in the works of Yu. A. Mitropol'skiy and his school (refs. 52-54, 44-46, 55). The method developed by Mitropol'skiy makes it possible to carry out, rather effectively, the investigation of nonstationary processes in nonlinear oscillatory systems which occur when frequencies, masses and other parameters of the nonlinear system vary. It should be pointed out that the scope of application for this method is rather wide, since its basic condition is a requirement that the variation in the parameters of the system be slow relative to "natural periods" of oscillation which is, in fact, satisfied in many problems.

We must also mention the works of Y. Z. Shtokalo (refs. 116-118) whose method, based on the asymptotic methods of Krylov and Bogolyubov, makes it possible to determine the criteria of stability and instability for solutions of linear differential equation systems whose coefficients are almost constant.

We should also make reference to the method of investigating the asymptotic behavior of solutions to linear differential equations developed by I. M. Rapoport (ref. 61). The basic idea of this method is the transformation /8
of a given system of differential equations into a special form which the author has called the L-diagonal form.

By using this method Rapoport has obtained a series of interesting results which pertain to the asymptotic properties of solutions to ordinary linear differential equations and to problems of the stability of motion.

All of the above investigations on the asymptotic theory of linear differential equations containing a parameter refer to the case when the roots of the characteristic equation remain simple or retain their multiplicity (in systems of the second order (ref. 76)) over the entire range of the argument. A rather important and interesting question concerning the asymptotic representation of solutions for the case when the multiplicity of the characteristic equation's roots varied at individual points over the range of argument variation, remained uninvestigated.

This gap was filled in a series of works by the authors of the present book (refs. 82-100, 104-115) as well as in the fundamental investigations of Yu. L. Daletskiy and S. G. Kreyn, (refs. 18-22). In this connection we should also mention the results obtained by A. G. Ilyukhin (refs. 29-31), I. I. Kovtun (refs. 34-38), I. I. Markush (refs. 50-51), A. A. Stonitskiy (refs. 72-75) et al. The above works, which were published during the period 1947 to 1965, have served as the basis for the present monograph.

The monograph consists of six chapters.

The first two chapters consider the question of the asymptotic representation of solutions to linear differential equations with free terms of the type $p(\tau, \varepsilon)e^{i\theta(t, \varepsilon)}$, where $p(\tau, \varepsilon)$ is a slowly varying function (or a vector function), while $\theta(t, \varepsilon)$ is a scalar function whose derivative has the form

$$\frac{d\theta}{dt} = k(\tau).$$

In this connection we distinguish between two cases:

(1) the case of "resonance" - when for some values of the argument from $[0, L]$, the function $ik(\tau)$ becomes equal to one (or several) of the roots of the corresponding characteristic equation;

(2) the case of "nonresonance" - when the values of $ik(\tau)$ do not coincide with any of the roots of the characteristic equation on $[0, L]$.

For each of the specified cases a different method is proposed for constructing the asymptotic solutions of the given differential equations.

The methods which have been developed prove to be rather effective in solving a series of practical problems. In the present monograph these methods are used to integrate differential equations with variable coefficients which describe the motion of mine-shaft lift cables with a load at the end.

The third chapter describes an asymptotic method for partitioning a system of linear differential equations into several independent subsystems of lower orders whose number depends on the number of isolated root groups for the characteristic equation. In the case of simple roots, the equations which have been partitioned are integrated by quadrature, i.e., we are able to obtain asymptotic solutions for the system investigated. /9

A very interesting but complicated case of constructing the asymptotic solutions for a system of linear differential equations is covered in Chapter 4.

This chapter considers the case when the roots of the characteristic equation contain multiple roots with multiple elementary dividers and when the "external frequency"--$ik(\tau)$--becomes equal to one of such (multiple) roots at some points in the interval considered.

The theory presented is applied to the investigation of differential equations with a small parameter for the higher derivatives.

The fifth chapter presents the results of those works of Yu. L. Daletskiy and S. G. Kreyn (refs. 18-22) in which the asymptotic methods are used to investigate differential equations in spaces of infinite dimensions.

In Chapter 6 an attempt is made to apply the asymptotic method described in the preceding chapters to the solution of certain problems associated with partial differential equations. In this connection the object in mind is to obtain computation formulas which are most simple and convenient to use.

At the end of the book there is a list of references which are concerned with the asymptotic representation of solutions to differential equations containing a parameter. We should point out that in the brief historical survey devoted to this problem we were unable to discuss many results which are interesting both from the standpoint of theory (refs. 23-25, 47, 48, 5-8), as well as from the standpoint of engineering applications (refs. 63, 67).

In conclusion the author expresses his sincere gratitude to Academician Nikolay Nikolayevich Bogolyubov, whose valuable advice was responsible to a large measure for the development of these methods.

The author also expresses his gratitude to Academicians of the Ukrainian Academy of Sciences Yu. A. Mitropol'skiy and I. Z. Shtokalo for a series of useful remarks which improved the book and to Doctor of Physical and Mathematical Sciences Yu. L. Daletskiy, who agreed to become the responsible editor of this monograph.

We are also grateful to Candidates of Physical and Mathematical Sciences A. G. Ilyukhin and I. I. Kovtun, who wrote Section 18 (A. G. Ilyukhin) and Sections 11, 27 (I. I. Kovtun).

CHAPTER 1. CONSTRUCTION OF THE ASYMPTOTIC SOLUTION FOR LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER WITH SLOWLY VARYING COEFFICIENTS

Section 1. Formulation of the Problem

The simplest example of differential equations which can be solved by the asymptotic method presented below is the equation /10

$$a(\tau, \varepsilon)\frac{d^2x}{dt^2} + \varepsilon c(\tau, \varepsilon)\frac{dx}{dt} + b(\tau, \varepsilon)x = p(\tau, \varepsilon)e^{i\theta(t,\varepsilon)}, \tag{1.1}$$

where $a(\tau, \varepsilon)$, $c(\tau, \varepsilon)$, $b(\tau, \varepsilon)$, $p(\tau, \varepsilon)$ are slowly varying functions which can be expanded as follows:

$$\begin{aligned} a(\tau, \varepsilon) &= \sum_{s=0}^{\infty} \varepsilon^s a_s(\tau), \qquad c(\tau, \varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s c_s(\tau), \\ b(\tau, \varepsilon) &= \sum_{s=0}^{\infty} \varepsilon^s b_s(\tau), \qquad p(\tau, \varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s p_s(\tau). \end{aligned} \tag{1.2}$$

Here τ is the so-called "slow" time which is determined by the relationship

$$\tau = \varepsilon t, \tag{1.3}$$

while ε is a small real parameter in the interval

$$0 < \varepsilon \leqslant \varepsilon_0. \tag{1.4}$$

We agree to call

$$a_0(\tau)\lambda^2(\tau) + b_0(\tau) = 0 \tag{1.5}$$

the characteristic equation for the differential equation (1.1).

In the future we shall assume that for all $\tau \in [0, L]$ the following conditions are satisfied:

$$a_0(\tau) \neq 0, \qquad \frac{b_0(\tau)}{a_0(\tau)} > 0, \qquad \frac{d\theta(t, \varepsilon)}{dt} = k(\tau)^*. \tag{1.6}$$

---

*Conditions (1.6) are necessitated not by the substance of the method presented below but rather by the isolation of the class of differential equations which describe oscillatory processes.

Therefore it follows from equation (1.5) that /11

$$\lambda_{1,2}(\tau) = \pm i \sqrt{\frac{b_0(\tau)}{a_0(\tau)}}. \tag{1.7}$$

Two cases can occur, depending on the value of the functions ik(τ) and of the roots (1.7):

(1) "the resonance case"--the function ik(τ) becomes equal to one of the roots of equation (1.5), for example to the root $\lambda_1(\tau)$, at certain points of the segment [0,L];

(2) "the nonresonance case"--when

$$ik(\tau) \neq \lambda_{1,2}(\tau) \tag{1.8}$$

for any $\tau \in [0, L]$.

In this chapter each of the cases will be considered separately.

We should point out that at this time we are considering, generally speaking, the Cauchy problem, i.e., we are seeking the solution of equation (1.1), which satisfies the initial conditions

$$x(t)|_{t=0} = x_0, \qquad \left.\frac{dx}{dt}\right|_{t=0} = x_0. \tag{1.9}$$

However, the method which is presented can also be applied to the solution of boundary value problems.

Section 2. Formal Solution in the Case of "Resonance"

In the case of "resonance" the following theorem can be proved.

Theorem 1.1. If the coefficients of equation (1.1) and the function k(τ) are infinitely differentiable with respect to τ on the segment [0,L], the formal particular solution of differential equation (1.1) may be represented in the form

$$x(t, \varepsilon) = \xi(t, \varepsilon) e^{i\theta(t,\varepsilon)}, \tag{2.1}$$

where the function $\xi(t, \varepsilon)$ satisfies the differential equation

$$\frac{d\xi}{dt} = [D(\tau, \varepsilon) + i(\Omega(\tau, \varepsilon) - k(\tau))]\xi + z(t, \varepsilon), \tag{2.2}$$

in which the coefficients allow formal expansions:

$$D(\tau, \varepsilon) = \sum_{s=1}^{\infty} \varepsilon^s D_s(\tau), \qquad \Omega(\tau, \varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s \Omega_s(\tau),$$
$$z(\tau, \varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s z_s(\tau). \tag{2.3}$$

The proof of this theorem will consist of determining the terms of expansion (2.3) in such a way that expression (2.1), in which $\xi(t, \varepsilon)$ is an integral of equation (2.2), formally satisfies equation (1.1). /12

For this purpose we shall substitute the function $x(t, \varepsilon)$, which is determined by relationships (2.1) and (2.2), into equation (1.1). We obtain the identity

$$\begin{aligned} &\{a(\tau, \varepsilon)[(D(\tau, \varepsilon) + i\Omega(\tau, \varepsilon))^2 + \varepsilon(D(\tau, \varepsilon) + i\Omega(\tau, \varepsilon))'] + \\ &+ \varepsilon c(\tau, \varepsilon)[D(\tau, \varepsilon) + i\Omega(\tau, \varepsilon)] + b(\tau, \varepsilon)\}\xi(t, \varepsilon) + a(\tau, \varepsilon)\{D(\tau, \varepsilon) + \\ &+ i[\Omega(\tau, \varepsilon) + k(\tau)] + z(\tau, \varepsilon) + \varepsilon z'(\tau, \varepsilon)\} + \varepsilon c(\tau, \varepsilon) = p(\tau, \varepsilon) \end{aligned} \tag{2.4}$$

(here and in the future the prime (') designates differentiation with respect to τ).

We equate separately the coefficients of $\xi(t, \varepsilon)$ and the free terms from both sides of this identity. As a result we obtain two relationships:

$$\begin{aligned} &a(\tau, \varepsilon)\{[D(\tau, \varepsilon) + i\Omega(\tau, \varepsilon)]^2 + \varepsilon[D(\tau, \varepsilon) + i\Omega(\tau, \varepsilon)]'\} + \\ &+ \varepsilon c(\tau, \varepsilon)[D(\tau, \varepsilon) + i\Omega(\tau, \varepsilon)] + b(\tau, \varepsilon) = 0, \end{aligned} \tag{2.5}$$

$$\begin{aligned} &a(\tau, \varepsilon)\{[D(\tau, \varepsilon) + i(\Omega(\tau, \varepsilon) + k(\tau))]z(\tau, \varepsilon) + \varepsilon z'(\tau, \varepsilon)\} + \\ &+ \varepsilon c(\tau, \varepsilon) z(\tau, \varepsilon) = p(\tau, \varepsilon), \end{aligned} \tag{2.6}$$

which make it possible for us to determine the coefficients of the formal series (2.3).

First we make use of relationship (2.5). By isolating in this relationship the coefficients of the identical powers of parameter ε, we obtain the following recurrence formulas:

$$-a_0(\tau)[\Omega_0(\tau)]^2 + b_0(\tau) = 0, \tag{2.7}$$

$$\begin{aligned} &\sum_{j=0}^{s} \sum_{m=0}^{s-j} a_j(\tau)[D_m(\tau) + i\Omega_m(\tau)][D_{s-j-m}(\tau) + i\Omega_{s-j-m}(\tau)] + \\ &+ \sum_{j=0}^{s-1} a_j(\tau)[D_{s-1-j}(\tau) + i\Omega_{s-1-j}(\tau)]' + \\ &+ \sum_{j=0}^{s-1} c_j(\tau)[D_{s-1-j}(\tau) + i\Omega_{s-1-j}(\tau)] + b_s(\tau) = 0, \qquad s = 1, 2, \ldots \end{aligned} \tag{2.8}$$

(in view of (2.3), $D_0 \equiv 0$).

From equation (2.7) we find

$$\Omega_0(\tau) = \pm \sqrt{\frac{b_0(\tau)}{a_0(\tau)}}. \tag{2.9}$$

In the future, while developing the particular solution of the nonhomogeneous equation (1.1), we shall take an arithmetical value for $\Omega_0(\tau)$. /13

Setting s=1 in equation (2.8), we obtain

$$\begin{aligned} 2i\Omega_0(\tau)\,a_0(\tau)\,[D_1(\tau) + i\Omega_1(\tau)] - a_1(\tau)\,\Omega_0^2(\tau) + ia_0(\tau)\,\Omega_0'(\tau) + \\ + i\Omega_0(\tau)c_0(\tau) + b_1(\tau) = 0, \end{aligned} \tag{2.10}$$

from which it follows that

$$\begin{aligned} D_1(\tau) &= -\frac{a_0(\tau)\,\Omega_0'(\tau) + c_0(\tau)\,\Omega_0(\tau)}{2\Omega_0(\tau)\,a_0(\tau)}, \\ \Omega_1(\tau) &= \frac{b_1(\tau) - a_1(\tau)\,\Omega_0^2(\tau)}{2\Omega_0(\tau)\,a_0(\tau)}. \end{aligned} \tag{2.11}$$

In the same manner we find the recurrence relationships for determining the functions $D_s(\tau)$ and $\Omega_s(\tau)$ when $s \geq 2$, specifically:

$$\begin{aligned} D_s(\tau) + i\Omega_s(\tau) = -\frac{1}{2ia_0(\tau)\,\Omega_0(\tau)} \Bigg\{ -a_0(\tau) \sum_{j=1}^{s-1} [D_j(\tau) + i\Omega_j(\tau)] \times \\ \times [D_{s-j}(\tau) + i\Omega_{s-j}(\tau)] + \sum_{j=1}^{s} \sum_{m=0}^{s-j} a_j(\tau)\,[D_m(\tau) + i\Omega_m(\tau)] \times \\ \times [D_{s-j-m}(\tau) + i\Omega_{s-j-m}(\tau)] + \sum_{j=0}^{s-1} a_j(\tau)\,[D_{s-1-j}(\tau) + i\Omega_{s-1-j}(\tau)]' + \\ + \sum_{j=0}^{s-1} c_j(\tau)\,[D_{s-1-j}(\tau) + i\Omega_{s-1-j}(\tau)] + b_s(\tau) \Bigg\}. \end{aligned} \tag{2.12}$$

Separating real and imaginary parts, we find the unknown functions $D_s(\tau)$ and $\Omega_s(\tau)$. From the proposed theorem and from equations (2.9), (2.11) and (2.12), it follows that the functions $D_s(\tau)$ and $\Omega_s(\tau)$ (s=0, 1,...) can be differentiated with respect to τ on the segment $[0, L]$.

To complete the proof of Theorem 1.1 we must determine the function $z(\tau, \varepsilon)$. For this purpose we use relationship (2.6). From the equality of coefficients of the same powers of parameter ε, we have

$$ia_0(\tau)[\Omega_0(\tau)+k(\tau)]z_0(\tau)=p_0(\tau), \tag{21.3}$$

$$ia_0(\tau)[\Omega_0(\tau)+k(\tau)]z_s(\tau)+\sum_{j=1}^{s}\sum_{j_1=1}^{s-j+1}a_j(\tau)[D_{j_1}(\tau)+i\Omega_{j_1}(\tau)]z_{s-j-j_1}(\tau)+$$

$$+a_0(\tau)\sum_{j=1}^{s}[D_j(\tau)+i\Omega_j(\tau)]z_{s-j}(\tau)+\sum_{j=0}^{s-1}a_j(\tau)z'_{s-1-j}(\tau)+$$

$$+\sum_{j=0}^{s-j}c_j(\tau)z_{s-j}(\tau)=p_s(\tau), \qquad s=1,\ 2,\ \ldots, \tag{2.14}$$

from which it follows that

$$z_0(\tau)=\frac{p_0(\tau)}{ia_0(\tau)[\Omega_0(\tau)+k(\tau)]},\quad z_s(\tau)=\frac{1}{ia_0(\tau)[\Omega_0(\tau)+k(\tau)]}\left\{p_s(\tau)-a_0(\tau)\sum_{j=1}^{s}[D_j(\tau)+i\Omega_j(\tau)]z_{s-j}(\tau)-\right. \tag{2.15}$$

$$\left.-\sum_{j=1}^{s}\sum_{j_1=0}^{s-j}a_j(\tau)[D_{j_1}(\tau)+i\Omega_{j_1}(\tau)]z_{s-j-j_1}(\tau)-\sum_{j=0}^{s-1}a_j(\tau)z'_{s-1-j}(\tau)+\sum_{j=0}^{s-1}c_j(\tau)z_{s-j}(\tau)\right\}, \qquad s=1,\ 2,\ \ldots. \tag{2.16}$$

Thus, having indicated a method for determining the coefficients of series (2.3), we have proved Theorem 1.1.

Note. Theorem 1.1 makes it possible to construct a particular solution of the nonhomogeneous equation (1.1).

As we know, to construct its general solution it is necessary to add the general solution of the following homogeneous equation to the particular solution which has been obtained:

$$a(\tau,\varepsilon)\frac{d^2x}{dt^2}+\varepsilon c(\tau,\varepsilon)\frac{dx}{dt}+b(\tau,\varepsilon)x(t,\varepsilon)=0. \tag{2.17}$$

The formal particular solution of this equation should be sought in the form

$$x(t,\varepsilon)=\xi(t,\varepsilon),$$

where

$$\frac{d\xi}{dt}=[D(\tau,\varepsilon)+i\Omega(\tau,\varepsilon)]\xi.$$

Here $D(\tau, \varepsilon)$ and $\Omega(\tau, \varepsilon)$ are constructed in the same way as in Theorem 1.1.

In this case, according to (2.9) we obtain two different values for

$\Omega_s(\tau)$ $(s=0, 1, \ldots)$, which makes it possible for us to construct two particular linearly independent solutions of equation (2.17), and consequently its general solution.

Section 3. Formal Solution in the Case of "Nonresonance"

Theorem 1.2. If the coefficients of equation (1.1) and the function $k(\tau)$ have derivatives of all orders with respect to τ on the segment $(0, L)$, the formal general solution of differential equation (1.1) in the case of "nonresonance" may be represented in the form /15

$$x(t, \varepsilon) = X(t, \varepsilon) + F(\tau, \varepsilon) e^{i\theta(t,\varepsilon)}, \tag{3.1}$$

where $X(t, \varepsilon)$ is the general solution of the homogeneous equation (2.17), while $F(\tau, \varepsilon)$ is a function which can be formally expanded as

$$F(\tau, \varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s F_s(\tau). \tag{3.2}$$

Proof. Since the method of determining the general solution $X(t, \varepsilon)$ is described in the Note of Section 2, it is only necessary to indicate the method of constructing the coefficients of series (3.2) to prove the given theorem. With this in mind we make the substitution

$$x(t, \varepsilon) = F(\tau, \varepsilon) e^{i\theta(t,\varepsilon)} \tag{3.3}$$

in equation (1.1). As a result we obtain

$$\begin{aligned} a(\tau, \varepsilon)[\varepsilon^2 F''(\tau, \varepsilon) + 2\varepsilon F'(\tau, \varepsilon) + \varepsilon i k'(\tau) F(\tau, \varepsilon) - k^2(\tau) F(\tau, \varepsilon)] + \\ + \varepsilon c(\tau, \varepsilon)[\varepsilon F'(\tau, \varepsilon) + ik(\tau) F(\tau, \varepsilon)] + b(\tau, \varepsilon) F(\tau, \varepsilon) = p(\tau, \varepsilon). \end{aligned} \tag{3.4}$$

By comparing the coefficients of the same powers of parameter ε in relationship (3.4), we obtain

$$a_0(\tau)\left[\frac{b_0(\tau)}{a_0(\tau)} - k^2(\tau)\right] F_0(\tau) = p_0(\tau), \tag{3.5}$$

$$a_0(\tau)\left[\frac{b_0(\tau)}{a_0(\tau)} - k^2(\tau)\right] F_s(\tau) = H_s(\tau), \qquad s = 1, 2, \ldots, \tag{3.6}$$

where

$$H_s(\tau) = p_s(\tau) - \sum_{j=0}^{s-2} a_j(\tau) F_{s-2-j}(\tau) - 2\sum_{j=0}^{s-1} a_j(\tau) F'_{s-1-j}(\tau) -$$
$$- ik'(\tau) \sum_{j=0}^{s-1} a_j(\tau) F_{s-1-j}(\tau) - k^2(\tau) \sum_{j=1}^{s} a_j(\tau) F_{s-j}(\tau) -$$
$$- \sum_{j=0}^{s-2} c_j(\tau) F'_{s-2-j}(\tau) - ik(\tau) \sum_{j=0}^{s-1} c_j(\tau) F_{s-1-j}(\tau) - \quad (3.7)$$
$$- \sum_{j=1} b_j(\tau) F_{s-j}(\tau), \qquad s = 1, 2, \ldots .$$

Now by taking into account conditions (1.5) and (1.6) we find /16

$$F_0(\tau) = \frac{H_0(\tau)}{b_0(\tau) - a_0(\tau) k^2(\tau)},$$
$$F_s(\tau) = \frac{H_s(\tau)}{b_0(\tau) - a_0(\tau) k^2(\tau)}, \qquad s = 1, 2, \ldots . \quad (3.8)$$

Theorem 1.2 has been proven.

Section 4. The Asymptotic Nature of the Solution

In preceding sections we have described a method of constructing a solution which formally satisfies equation (1.1) for cases of "resonance" and "nonresonance"; in the present section we shall show that a solution constructed in this manner has an asymptotic nature.

Let us consider in detail the case of "resonance."

We examine the function

$$x^{(m)}(t, \varepsilon) = \xi^{(m)}(t, \varepsilon) e^{i\vartheta(t,\varepsilon)}, \quad (4.1)$$

where $\xi^{(m)}(t, \varepsilon)$ is determined by the differential equation

$$\frac{d\xi^{(m)}}{dt} = [D^{(m)}(\tau, \varepsilon) + i(\Omega^{(m)}(\tau, \varepsilon) - k(\tau))]\xi^{(m)} + z^{(m)}(t, \varepsilon), \quad (4.2)$$

in which

$$D^{(m)}(\tau, \varepsilon) = \sum_{s=1}^{m} \varepsilon^s D_s(\tau), \qquad \Omega^{(m)}(\tau, \varepsilon) = \sum_{s=0}^{m} \varepsilon^s \Omega_s(\tau),$$
$$z^{(m)}(\tau, \varepsilon) = \sum_{s=0}^{m} \varepsilon^s z_s(\tau). \quad (4.3)$$

In the future we shall refer to the function $x^{(m)}(t, \varepsilon)$ (m is a natural number) as the m-th approximation of the solution to equation (1.1).

Taking into account equation (4.2) and substituting the value of $x^{(m)}(t, \varepsilon)$ into expression

$$a(\tau, \varepsilon)\frac{d^2x^{(m)}}{dt^2} + \varepsilon c(\tau, \varepsilon)\frac{dx^{(m)}}{dt} + b(\tau, \varepsilon)x^{(m)} - p(\tau, \varepsilon)e^{i\theta(t,\varepsilon)},$$

we obtain

$$\begin{aligned} &a(\tau, \varepsilon)\frac{d^2x^{(m)}}{dt^2} + \varepsilon c(\tau, \varepsilon)\frac{dx^{(m)}}{dt} + b(\tau, \varepsilon)x^{(m)}(t, \varepsilon) - p(\tau, \varepsilon)e^{i\theta(t,\varepsilon)} = \\ &= \{a(\tau, \varepsilon)[(D^{(m)}(\tau, \varepsilon) + i\Omega^{(m)}(\tau, \varepsilon))^2 + \varepsilon(D^{(m)}(\tau, \varepsilon) + i\Omega^{(m)}(\tau, \varepsilon))'] + \\ &+ \varepsilon c(\tau, \varepsilon)[D^{(m)}(\tau, \varepsilon) + i\Omega^{(m)}(\tau, \varepsilon)] + b^{(m)}(\tau, \varepsilon)\}\xi^{(m)}(t, \varepsilon)e^{i\theta(t,\varepsilon)} + \\ &+ \{a(\tau, \varepsilon)[(D^{(m)}(\tau, \varepsilon) + i\Omega^{(m)}(\tau, \varepsilon) + ik(\tau))z^{(m)}(\tau, \varepsilon) + \\ &+ \varepsilon z^{(m)'}(\tau, \varepsilon)] + \varepsilon c(\tau, \varepsilon)z^{(m)}(\tau, \varepsilon) - p(\tau, \varepsilon)\}e^{i\theta(t,\varepsilon)}. \end{aligned} \tag{4.4}$$

According to the construction of quantities contained in the m-th approximation, the functions in the right side of relationship (4.4) which are in front of $\xi^{(m)}(t, \varepsilon)e^{i\theta(t, \varepsilon)}$ and in front of $e^{i\theta(t, \varepsilon)}$ are infinitesimals of order ε^{m+1}. Therefore we can write

$$\begin{aligned} &a(\tau, \varepsilon)\frac{d^2x^{(m)}}{dt^2} + \varepsilon c(\tau, \varepsilon)\frac{dx^{(m)}}{dt} + b(\tau, \varepsilon)x^{(m)}(t, \varepsilon) = \\ &= p(\tau, \varepsilon)e^{i\theta(t,\varepsilon)} + \varepsilon^{m+1}[r_m(\tau, \varepsilon)\xi^{(m)}(t, \varepsilon) + f_m(\tau, \varepsilon)], \end{aligned} \tag{4.5}$$

where $r_m(\tau, \varepsilon)$, $f_m(\tau, \varepsilon)$ are functions which are regular with respect to ε in the neighborhood of the point $\varepsilon=0$.

Integrating equation (4.2), we find

$$\begin{aligned} \xi^{(m)}(t, \varepsilon) = &Ae^{\int_0^t [D^{(m)}(\tau,\varepsilon)+i(\Omega^{(m)}(\tau,\varepsilon)-k(\tau))]dt} + \\ &+ \int_0^t z^{(m)}(\sigma, \varepsilon)e^{\int_s^t [D^{(m)}(\tau,\varepsilon)+i(\Omega^{(m)}(\tau,\varepsilon)-k(\tau))]dt} ds, \end{aligned} \tag{4.6}$$

where A is the integration constant, $\sigma = \varepsilon s$.

Since the functions $D^{(m)}(\tau, \varepsilon), z^{(m)}(\tau, \varepsilon)$ are differentiable with respect to τ over the interval $[0, L]$, they are bounded for any ε from (1.4).

Consequently we can specify such constants M_1 and M_2, which are independent of ε, so that

$$\left|\sum_{j=1}^{m}\varepsilon^{j-1}D_j(\tau)\right| \leqslant M_1, \qquad |z^{(m)}(\tau,\varepsilon)| \leqslant M_2. \tag{4.7}$$

By using these inequalities and (4.6) we can estimate $\xi^{(m)}(t,\varepsilon)$:

$$|\xi^{(m)}(t,\varepsilon)| \leqslant Ae^{M_1L} + M_2e^{M_1L}t \leqslant \left(A + \frac{M_2L}{\varepsilon}\right)e^{M_1L}. \tag{4.8}$$

Now equation (4.5) can be written in the form

$$a(\tau,\varepsilon)\frac{d^2x^{(m)}}{dt^2} + \varepsilon c(\tau,\varepsilon)\frac{dx^{(m)}}{dt} + b(\tau,\varepsilon)x^{(m)}(t,\varepsilon) = p(\tau,\varepsilon)e^{i\theta(t,\varepsilon)} + \varepsilon^m g_m(\tau,\varepsilon), \tag{4.9}$$

where $g_m(\tau,\varepsilon)$ is a function which is bounded for all $\tau\in[0,L]$ and $0<\varepsilon\leq\varepsilon_0$. /18

Thus the m-th approximation satisfies the initial equation with an accuracy determined by quantities which are infinitesimals of order ε^m, uniformly with respect to t on the segment $0\leq t\leq\frac{L}{\varepsilon}$.

Theorem 1.3. If the conditions of Theorem 1.1 are satisfied and if

$$x(t,\varepsilon)|_{t=0} = x^{(m)}(t,\varepsilon)|_{t=0} = x_0, \left.\frac{dx}{dt}\right|_{t=0} = \left.\frac{dx^{(m)}}{dt}\right|_{t=0} = x_0, \tag{4.10}$$

for any $L>0$ and $0<\varepsilon\leq\varepsilon_0$, we can specify a constant C which is independent of ε, so that the following estimates will be valid:

$$|x(t,\varepsilon) - x^{(m)}(t,\varepsilon)| \leqslant \varepsilon^{m-1}C, \qquad \left|\frac{dx}{dt} - \frac{dx^{(m)}}{dt}\right| \leqslant \varepsilon^{m-1}C. \tag{4.11}$$

Before proceeding with the proof of the above theorem we prove two auxiliary lemmas.

Lemma 1.1. If for all $t\in\left|0,\frac{L}{\varepsilon}\right|$ the inequality

$$r(t) \leqslant c\int_0^t r(s)\,ds + f(t), \tag{4.12}$$

is satisfied, where c is a positive constant while the function f(t) is differentiable on the segment considered,

$$r(t) \leqslant f(0)\, e^{ct} + \int_0^t e^{c(t-s)} f'(s)\, ds. \tag{4.13}$$

The proof of the above lemma is taken from reference 39. Let us assume that

$$R(t) = \int_0^t r(s)\, ds. \tag{4.14}$$

Then inequality (4.12) may be rewritten in the form

$$R'(t) \leqslant cR(t) + f(t). \tag{4.15}$$

Multiplying both sides of inequality (4.15) by e^{-ct} and integrating the results from 0 to t, we obtain

$$R(t) \leqslant \int_0^t e^{c(t-s)} f(s)\, ds. \tag{4.16}$$

Integrating by parts, we represent inequality (4.16) in the form /19

$$R(t) \leqslant \frac{1}{c}\left[\int_0^t e^{c(t-s)} f'(s)\, ds + e^{ct} f(0) - f(t)\right]. \tag{4.17}$$

Now in accordance with inequalities (4.14) and (4.17) we can write the initial inequality (4.12) in the form (4.13), which was to be proven.

Lemma 1.2. If $y(t, \varepsilon)$ is a solution of the differential equation

$$\frac{d^2y}{dt^2} + \omega^2(\tau, \varepsilon) y = q(\tau, \varepsilon) \tag{4.18}$$

with initial conditions

$$y(0) = \left.\frac{dy}{dt}\right|_{t=0} = 0, \tag{4.19}$$

where

$$\omega(\tau, \varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s \omega_s(\tau), \qquad \omega_0(\tau) \neq 0, \qquad \tau \in [0, L], \tag{4.20}$$

any $L>0$ will have a corresponding constant S_1 which is independent of ε, such that on the segment $0 \leqslant t \leqslant \frac{L}{\varepsilon}$ the following inequality exists:

$$|y(t,\varepsilon)| \leqslant 2e^{2S_1 L}\int_0^t \left|\frac{q(\tau,\varepsilon)}{\omega(\tau,\varepsilon)}\right| dt,$$

$$\left|\frac{dy}{dt}\right| \leqslant 2\omega^* e^{2S_1 L}\int_0^t \left|\frac{q(\tau,\varepsilon)}{\omega(\tau,\varepsilon)}\right| dt, \qquad (4.21)$$

$$\omega^* = \max_{\substack{0\leqslant\tau\leqslant L\\ 0<\varepsilon\leqslant\varepsilon_0}} \omega(\tau,\varepsilon).$$

Proof. We introduce new variables u and v by means of equations

$$\begin{aligned} y &= u\cos\theta_1 + v\sin\theta_1,\\ \frac{dy}{dt} &= -\omega(\tau,\varepsilon)u\sin\theta_1 + \omega(\tau,\varepsilon)v\cos\theta_1, \end{aligned} \qquad (4.22)$$

where

$$\theta_1 = \int_0^t \omega(\tau,\varepsilon)\,dt. \qquad (4.23)$$

Then equation (4.18) may be transformed into an equivalent system of first-order equations of the form /20

$$\begin{aligned} \frac{du}{dt} &= q_1(\tau,\varepsilon) + \varepsilon[a(\tau,\varepsilon)u + b(\tau,\varepsilon)v],\\ \frac{dv}{dt} &= q_2(\tau,\varepsilon) + \varepsilon[a_1(\tau,\varepsilon)u + b_1(\tau,\varepsilon)v], \end{aligned} \qquad (4.24)$$

where

$$\begin{aligned} a(\tau,\varepsilon) &= -\frac{\omega'(\tau,\varepsilon)\sin^2\theta_1}{\omega(\tau,\varepsilon)}, \qquad b_1(\tau,\varepsilon) = -\frac{\omega'(\tau,\varepsilon)\cos^2\theta_1}{\omega(\tau,\varepsilon)},\\ a_1(\tau,\varepsilon) = b(\tau,\varepsilon) &= \frac{\omega'(\tau,\varepsilon)\sin 2\theta_1}{2\omega(\tau,\varepsilon)}, \qquad q_1(\tau,\varepsilon) = \frac{q(\tau,\varepsilon)\sin\theta_1}{\omega(\tau,\varepsilon)}, \qquad (4.25)\\ q_2(\tau,\varepsilon) &= \frac{q(\tau,\varepsilon)\cos\theta_1}{\omega(\tau,\varepsilon)} \end{aligned}$$

Since according to condition (4.19),

$$u|_{t=0} = v|_{t=0} = 0, \qquad (4.26)$$

system (4.24) may be replaced with an equivalent integral system

$$u = \int_0^t [q_1(\tau, \varepsilon) + \varepsilon (a(\tau, \varepsilon) u + b(\tau, \varepsilon) v)]\, dt,$$
$$v = \int_0^t [q_2(\tau, \varepsilon) + \varepsilon (a_1(\tau, \varepsilon) u + b_1(\tau, \varepsilon) v)]\, dt, \tag{4.27}$$

from which it follows that

$$|u| \leqslant \int_0^t \left| \frac{q(\tau, \varepsilon)}{\omega(\tau, \varepsilon)} \right| dt + \varepsilon S_1 \int_0^t [|u| + |v|]\, dt,$$
$$|v| \leqslant \int_0^t \left| \frac{q(\tau, \varepsilon)}{\omega(\tau, \varepsilon)} \right| dt + \varepsilon S_1 \int_0^t [|u| + |v|]\, dt, \tag{4.28}$$

where

$$S_1 = \max \{|a(\tau, \varepsilon)|, |a_1(\tau, \varepsilon)|, |b_1(\tau, \varepsilon)|\},$$
$$\tau \in [0, L], \qquad \varepsilon \in (0, \varepsilon_0]. \tag{4.29}$$

By adding inequalities (4.28), we obtain /21

$$r \leqslant 2 \int_0^t \left| \frac{q(\tau, \varepsilon)}{\omega(\tau, \varepsilon)} \right| dt + 2\varepsilon S_1 \int_0^t r dt, \tag{4.30}$$

where

$$= |u(t, \varepsilon)| + |v(t, \varepsilon)|. \tag{4.31}$$

Inequality (4.30) is analogous to inequality (4.12); therefore, according to Lemma 1.1,

$$r \equiv |u| + |v| \leqslant 2 \int_0^t \left| \frac{q(\tau_1, \varepsilon)}{\omega(\tau_1, \varepsilon)} \right| e^{2\varepsilon S_1 (t - t_1)} dt_1, \qquad \tau_1 = \varepsilon t_1,$$

or

$$|u| + |v| \leqslant 2e^{2S_1 L} \int_0^t \left| \frac{q(\tau_1, \varepsilon)}{\omega(\tau_1, \varepsilon)} \right| dt_1. \tag{4.32}$$

The validity of Lemma 1.2 follows from inequality (4.32) and from equations (4.22). Let us now prove Theorem 1.3.

We let

$$y(t, \varepsilon) = x(t, \varepsilon) - x^{(m)}(t, \varepsilon), \tag{4.33}$$

where $x(t, \varepsilon)$ is the exact solution of equation (1.1), while $x^{(m)}(t, \varepsilon)$ is its m-th approximation. Then, according to (1.1) and (4.9), the function $y(t, \varepsilon)$ satisfies the equation

$$a(\tau, \varepsilon)\frac{d^2y}{dt^2} + \varepsilon c(\tau, \varepsilon)\frac{dy}{dt} + b(\tau, \varepsilon)y = -\varepsilon^m g(\tau, \varepsilon), \tag{4.34}$$

which may be represented in the form (4.18), where

$$\begin{gathered} q(\tau, \varepsilon) = -\frac{1}{a(\tau, \varepsilon)}\left[\varepsilon^m g(\tau, \varepsilon) + \varepsilon c(\tau, \varepsilon)\frac{dy}{dt} + \varepsilon b_1(\tau, \varepsilon)y\right], \\ b_1(\tau, \varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s b_{s+1}(\tau), \\ \omega^2(\tau, \varepsilon) = \frac{b_0(\tau)}{a(\tau, \varepsilon)}, \end{gathered} \tag{4.35}$$

and

$$y(0) = \left.\frac{dy}{dt}\right|_{t=0} = 0.$$

By using Lemma 1.2, which we have just proven, we obtain /22

$$\begin{gathered} |y(t, \varepsilon)| \leqslant S_2 \int_0^t \left|\frac{q(\tau, \varepsilon)}{\omega(\tau, \varepsilon)}\right| dt, \\ \left|\frac{dy}{dt}\right| \leqslant S_3 \int_0^t \left|\frac{q(\tau, \varepsilon)}{\omega(\tau, \varepsilon)}\right| dt, \end{gathered} \tag{4.36}$$

where

$$S_2 = 2e^{2S_1 L}, \qquad S_3 = \omega^* S_2. \tag{4.37}$$

According to (4.35), inequality (4.36) assumes the form

$$|y| \leqslant S_2 \int_0^t \left|\frac{1}{\sqrt{b_0(\tau)a(\tau, \varepsilon)}}\left[\varepsilon^m g_m(\tau, \varepsilon) + \varepsilon c(\tau, \varepsilon)\frac{dy}{dt} + b_1(\tau, \varepsilon)y\right]\right| dt, \tag{4.38}$$

$$\left|\frac{dy}{dt}\right| \leqslant S_3 \int_0^t \left|\frac{1}{\sqrt{b_0(\tau)\,a(\tau,\varepsilon)}}\right| \left[\varepsilon^m g_m(\tau,\varepsilon) + \varepsilon c(\tau,\varepsilon)\frac{dy}{dt} + b_1(\tau,\varepsilon)y\right] dt.$$

Since the functions $b_0(\tau)$, $a(\tau,\varepsilon)$, $g_m(\tau,\varepsilon)$, $c(\tau,\varepsilon)$ and $b_1(\tau,\varepsilon)$ are differentiable with respect to τ, we can specify such constants C_1 and C_2 for all $t\in\left[0,\frac{L}{\varepsilon}\right]$ and $\varepsilon(0<\varepsilon\leq\varepsilon_0)$ which are independent of ε, so that the following estimates will be valid:

$$\left|\frac{g_m(\tau,\varepsilon)}{\sqrt{b_0(\tau)\,a(\tau,\varepsilon)}}\right| \leqslant C_1, \qquad \left|\frac{c(\tau,\varepsilon)}{\sqrt{b_0(\tau)\,a(\tau,\varepsilon)}}\right| \leqslant C_2,$$
$$\left|\frac{b_1(\tau,\varepsilon)}{\sqrt{b_0(\tau)\,a(\tau,\varepsilon)}}\right| \leqslant C_2. \tag{4.39}$$

Consequently inequalities (4.38) can be rewritten in the form

$$|y(t,\varepsilon)| \leqslant \varepsilon S_2 C_2 \int_0^t \left(|y| + \left|\frac{dy}{dt}\right|\right) dt_1 + \varepsilon^m S_3 C_1 t,$$
$$\left|\frac{dy}{dt}\right| \leqslant \varepsilon S_3 C_2 \int_0^t \left(|y| + \left|\frac{dy}{dt}\right|\right) dt_1 + \varepsilon^m S_3 C_1 t. \tag{4.40}$$

By adding the resulting inequalities and introducing the designation /23

$$W = |y| + \left|\frac{dy}{dt}\right|, \tag{4.41}$$

we obtain the inequality

$$W \leqslant \varepsilon K_1 \int_0^t W dt + \varepsilon^{m-1} K_2, \tag{4.42}$$

in which

$$K_1 = C_2(S_2 + S_3), \qquad K_2 = C_1 L(S_1 + S_2). \tag{4.43}$$

In accordance with Lemma 1.1, we find from inequality (4.42) that

$$W \leqslant \varepsilon^{m-1} C, \quad \text{where} \quad C = K_2 e^{KL}, \tag{4.44}$$

and consequently

$$|y| \leqslant \varepsilon^{m-1} C, \left| \frac{dy}{dt} \right| \leqslant \varepsilon^{m-1} C. \tag{4.45}$$

The theorem has been proved.

It follows from the resulting inequalities (4.45) that the m-th approximation $x^{(m)}(t, \varepsilon)$ and its derivative $\frac{dx^{(m)}}{dt}$ approach respectively the exact solution $x(t, \varepsilon)$ and $\frac{dx}{dt}$ when $\varepsilon \to 0$ beginning with $m \geq 2$. In this way we have proved the asymptotic nature of the solution which we have constructed for the case of "resonance."

For the case of "nonresonance" we can carry out analogous reasoning and obtain the following estimate in line with the conditions of the theorem:

$$|x(t, \varepsilon) - x^{(m)}(t, \varepsilon)| \leqslant \varepsilon^m C, \qquad \left| \frac{dx}{dt} - \frac{dx^{(m)}}{dt} \right| \leqslant \varepsilon^m C, \tag{4.46}$$

where C is a constant which is independent of ε.

In the case when we have a small free term ($p_0(\tau) \equiv 0$), the asymptotic estimates also have the form (4.46) in the case of "resonance."

Note. Above we have presented an algorithm for the construction of the asymptotic solution for differential equation (1.1) with the right side $p(\tau, \varepsilon) e^{i\theta(t, \varepsilon)}$. The results which have been obtained may be transposed quite easily to the case when the right side in the differential equation (1.1) has the form

$$\sum_{j=1}^{N} p_j(\tau, \varepsilon) e^{i\theta_j(t, \varepsilon)}.$$

Section 5. Establishing the Forces in a Visco-Elastic Thread of Variable Length

As an example which illustrates the theory presented above we shall consider the problem of finding forces in a visco-elastic thread of variable length $l = l(t)$ with a load Q at the end. As shown in references 32 and 63, this problem can be reduced to the integration of equation /24

$$\frac{l}{g}\left(Q + \frac{1}{3} ql\right) \frac{d^2\varphi}{dt^2} + \left[\alpha + \frac{1}{g}\left(Q + \frac{ql}{2}\right)\frac{dl}{dt}\right] \frac{d\varphi}{dt} + K\varphi =$$

$$= \frac{1}{g}\left(Q + \frac{ql}{2}\right)\left(g - \frac{dV_c}{dt}\right), \tag{5.1}$$

where φ is the relative elongation of the thread, q is the weight per meter of the thread, α is a coefficient which characterizes the damping of dynamic

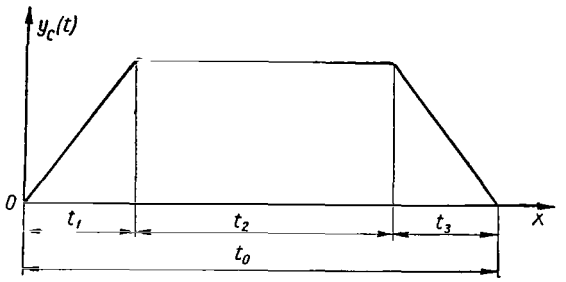

Figure 1.

forces in the thread, g is the acceleration due to gravity, v_c is the linear velocity of points on the circumference of a drum used to wind the thread; $K=E\sigma$, E is the modulus of elasticity for the thread, σ is the area of the transverse cross section of the thread.

In practice, the load Q is lifted most frequently by using a trapezoidal tachogram (fig. 1). (In regard to this, see reference 67, pages 48-50.)

Consequently, equation (5.1) must be integrated over each of the following three regions:

(1) The region of uniform acceleration:

$$l = l_0 - \frac{at^2}{2}, \qquad v_c = \frac{dl}{dt} = -at, \qquad \frac{dv_c}{dt} = -a; \tag{5.2}$$

(2) The region of uniform motion:

$$l = l_1 - v_0 t, \qquad v_c = \frac{dl}{dt} = -v_0, \qquad \frac{dv}{dt} = 0, \tag{5.3}$$

$$l_1 = l_0 - \frac{at_1^2}{2}, \qquad v_0 = at_1; \quad \text{/25}$$

(3) The region of uniform deceleration:

$$l = l_2 - v_0 t + \frac{at^2}{2}, \qquad v_c = \frac{dl}{dt} = -v_0 + at,$$

$$\frac{dv}{dt} = a, \qquad l_2 = l_1 - v_0(t_2 - t_1), \tag{5.4}$$

where l_0 is the initial length of the thread.

In order to apply the asymptotic method presented above to the integration of differential equation (5.1), the latter must be represented in the form (1.1) on each of the regions. To do this we follow reference 103 and consider the following quantities in the first weight-lifting region:

$$T = \omega_0 t, \qquad \omega_0 = \frac{Q^2}{\alpha q l_0}, \qquad \varepsilon = \sqrt[3]{\frac{a}{\omega_0^2 l_0}}. \tag{5.5}$$

Then equation (5.1) assumes the form

$$\begin{aligned} &\left|\left(\frac{Q}{l_0}+\frac{q}{3}\right)-\varepsilon\left(\frac{1}{2}\cdot\frac{Q}{l_0}+\frac{1}{3}q\right)\tau^2+\varepsilon^2\frac{q\tau^4}{12}\right|\frac{d^2\varphi}{dT^2}+ \\ &+\varepsilon\left\{\frac{\alpha g}{l_0^2}\sqrt[3]{\frac{\alpha q l_0^2}{aQ^2}}-\varepsilon\left|\left(\frac{Q}{l_0}+\frac{1}{2}q\right)\tau-\frac{q}{4}\sqrt[3]{\frac{\alpha^2 q^2 a l_0}{Q^4}}\tau^3\right|\right\}\frac{d\varphi}{dT}+ \\ &+\frac{Kg\alpha^2q^2}{Q^4}\varphi=\varepsilon\left|\left(\frac{Q}{l_0}+\frac{q}{2}\right)-\varepsilon\frac{q\tau^2}{4}\right|\left(1+\frac{g}{a}\right)\sqrt[3]{\frac{\alpha^4 q^4 l_0^2 a^2}{Q^8}}, \end{aligned} \tag{5.6}$$

where

$$\tau = \varepsilon T. \tag{5.7}$$

In the second region, for the movement of weight Q equation (5.1), after simple transformations, may be written in the following manner:

$$\begin{aligned} &\left[\frac{Q}{l_1}+\frac{1}{3}q-\varepsilon\left(\frac{Q}{l_1}+\frac{2}{3}q\right)\tau+\varepsilon^2\frac{q\tau^2}{3}\right]\frac{d^2\varphi}{dT^2}+ \\ &+\varepsilon\left\{\left[\frac{\alpha g}{l_1 v_0}-\left(\frac{Q}{l_1}+\frac{1}{2}q\right)\right]\sqrt{\frac{v_0}{l_1\omega_1}}+\varepsilon\frac{q}{2}\sqrt{\frac{v_0}{l_1\omega_1}}\tau\right\}\frac{d\varphi}{dT}+ \\ &+\frac{Kg}{l_1^2\omega_1^2}\varphi=\varepsilon\left[\frac{Q}{l_1}+\frac{q}{2}-\varepsilon\frac{q\tau}{2}\right]\frac{g}{v_0\omega_1}\sqrt{\frac{v_0}{l_1\omega_1}}, \end{aligned} \tag{5.8}$$

where /26

$$T = \omega_1 t, \qquad \omega_1 = \frac{Q^2}{\alpha g l_1}, \qquad \tau = \varepsilon T, \qquad \varepsilon = \sqrt{\frac{v_0}{l_1\omega_1}}, \tag{5.9}$$

and finally, in the third region,

$$\left\{\frac{Q}{l_2}+\frac{q}{3}-\varepsilon\left[\left(\frac{Q}{l_2}+\frac{2}{3}q\right)\tau-\sqrt{\frac{v_0}{l_1\omega_2}}\left(\frac{Q}{2v_0^2}+\frac{q}{3}+\frac{qal_2}{3v_0^2}\right)\tau^2\right]-\right.$$

$$-\varepsilon^2\left[\sqrt{\frac{v_0}{l_2\omega_2}}\frac{qal_2}{3v_0^2}\tau^3-\frac{qa^2l_2}{12v_0^3\omega_2}\tau^4\right]\Bigg\}\frac{d^2\varphi}{dT^2}+$$

$$+\varepsilon\left\{\sqrt{\frac{v_0}{l_2\omega_2}}\left|\frac{ag}{v_0l_2}-\left(\frac{Q}{l_2}+\frac{q}{2}\right)+\left(\frac{Qa}{v_0^2}+\frac{q}{2}+\frac{qal_2}{2v_0^2}\right)\sqrt{\frac{v_0}{l^2\omega^2}}\tau\right]-\right. \tag{5.10}$$

$$-\varepsilon\left[\frac{3aq}{4v_0\omega_2}\tau^2-\frac{qa^2l_2}{4v_0^2\omega_2}\sqrt{\frac{v_0}{l_2\omega_2}}\tau^3\right]\Bigg\}\frac{d\varphi}{dT}+\frac{Kg}{l_2^2\omega_2^2}\varphi=$$

$$=\varepsilon\left[\frac{Q}{l_2}+\frac{q}{2}-\varepsilon\left(\frac{q\tau}{2}-\frac{qal_2}{4v_0}\sqrt{\frac{v_0}{l_2\omega_2}}\tau^2\right]\frac{g-a}{v_0\omega_2}\sqrt{\frac{v_0}{l_2\omega_2}},\right.$$

where

$$T=\omega_2 t,\qquad \omega_2=\frac{Q^2}{\alpha q l_2},\qquad \tau=\varepsilon T,\qquad \varepsilon=\sqrt{\frac{v_0}{l_2\omega_2}}. \tag{5.11}$$

The resulting equations (5.6), (5.8) and (5.10) are equations of form (1.1) in which

$$\theta(t,\varepsilon)\equiv 0. \tag{5.12}$$

Consequently Theorem 1.2, which refers to the case of "nonresonance," is applicable to them.

By using the equations of Section 3 we can construct an asymptotic solution $\varphi^{(m)}$ for the above equations for any natural m. In particular, in the second approximation (m=2), the solution of equation (5.6) has the form

$$\varphi^{(2)}=Be^{-(\beta_1 t+\beta_2 t^2+\beta_3 t^3+\beta_4 t^4)}\cos(\delta_0+\delta_1 t+\delta_2 t^2+\delta_3 t^3+\delta_4 t^4)+ \tag{5.13}$$

$$+\frac{a+g}{Kg}\left(Q+\frac{ql_0}{2}-\frac{qat^2}{4}\right),$$

where

$$\beta_1=\frac{\alpha g}{2l_0\left(Q+\frac{ql_0}{3}\right)},\qquad \beta_2=-\frac{a}{8l_0},\qquad \beta_3=\frac{a\alpha g\left(\frac{Q}{2}+\frac{ql_0}{3}\right)}{6l_0^2\left(Q+\frac{ql_0}{3}\right)},$$

$$\beta_4=\frac{a^2g}{32l_0\left(Q+\frac{ql_0}{3}\right)},\qquad \delta_1=\nu_0-\frac{\alpha^2\nu_0^3}{8K^2},$$

$$\delta_2=\frac{a\nu_0^3}{6Kg}\left(\frac{KQ}{l_0\nu_0^2}-\frac{1}{2}Q\right),\qquad \nu_0=\sqrt{\frac{Kg}{l_0\left(Q+\frac{1}{3}ql_0\right)}}, \tag{5.14}$$

$$\delta_3 = \frac{a\nu_0^5}{40K^2g^2}\left| 3\left(\frac{Kg}{l_0\nu_0^2} - \frac{1}{2}Q\right)^2 - \frac{qKg}{3\nu_0^2}\right|,$$

B, δ_0 are constants of integration determined from the initial conditions. /27

We shall represent the solution of equation (5.8) in the form

$$\varphi^{(2)} = B_1 e^{-(\beta_{11}t+\beta_{21}t^2)}\cos(\delta_{01} + \delta_{11}t + \delta_{21}t^2 + \delta_{31}t^3) + \\ + \frac{1}{K}\left|Q + \frac{q}{2}(l_1 - v_0 t)\right|, \tag{5.15}$$

where

$$\beta_{11} = \frac{2ag - v_0\left(Q + \frac{1}{3}ql_1\right)}{4l_1\left(Q + \frac{1}{3}ql_1\right)},$$

$$\beta_{21} = \frac{v_0^2 g}{8l_1\left(Q + \frac{1}{3}ql_1\right)} + \frac{v_0\left(Q + \frac{2}{3}ql_1\right)\left[ag - v_0\left(Q + \frac{1}{2}ql_1\right)\right]}{4l_1^2\left(Q + \frac{1}{3}ql_1\right)^2},$$

$$\delta_{11} = \nu_{11} - \frac{\left[ag - v_0\left(Q + \frac{1}{2}ql_1\right)\right]^2\nu_{11}^3}{8K^2g^2},$$

$$\delta_{21} = \frac{v_0\left(Q + \frac{2}{3}ql_1\right)\nu_{11}^3}{4Kg}, \tag{5.16}$$

$$\delta_{31} = \frac{v_0\left(Q + \frac{2}{3}ql_1\right)^2\nu_{11}^5}{8K^2g^2} - \frac{v_0^2 g\nu_{11}^3}{18Kg},$$

$$\nu_{11} = \sqrt{\frac{Kg}{l_1\left(Q + \frac{1}{3}gl_1\right)}},$$

B_1, δ_{01} are constants of integration. /28

Finally, by solving equation (5.10), we obtain

$$\varphi^{(2)} = B_2 e^{-(\beta_{12}t+\beta_{22}t^2+\beta_{32}t^3+\beta_{42}t^4)}\cos(\delta_{02} + \delta_{12}t + \delta_{22}t^2 + \delta_{32}t^3 + \delta_{42}t^4 + \\ + \delta_{52}t^5) + \frac{g-a}{Kg}\left[Q + \frac{1}{2}q\left(l_2 - v_0 t + \frac{1}{2}at^2\right)\right], \tag{5.17}$$

where

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$$\beta_{12}=\frac{2\alpha g-v_0\left(Q+\frac{1}{3}ql_2\right)}{4l_2\left(Q+\frac{1}{3}ql_2\right)},$$

$$\beta_{22}=\frac{Qa+\frac{1}{3}qv_0^2+\frac{qal_2}{3}}{8l_2\left(Q+\frac{1}{3}ql_2\right)}+\frac{v_0\left(Q+\frac{2}{3}ql_2\right)\left[\alpha g-v_0\left(Q+\frac{ql_2}{2}\right)\right]}{4l_2^2\left(Q+\frac{1}{3}ql_2\right)^2},$$

$$\beta_{32}=\frac{v_0\left(Q+\frac{2}{3}ql_2\right)\left(Qa+\frac{qv_0^2}{2}+\frac{qal_2}{2}\right)}{6l_2^2\left(Q+\frac{ql_2}{3}\right)^2}-\frac{aqv_0}{8l_2\left(Q+\frac{ql_2}{3}\right)^2}-$$

$$-\frac{\left(\frac{1}{2}Qa+\frac{qv_0^2}{3}+\frac{qal_2}{3}\right)\left[\alpha g-v_0\left(Q+\frac{ql_2}{2}\right)\right]}{6l_2^2\left(Q+\frac{1}{3}ql_2\right)^2},$$

$$\beta_{42}=\frac{qa^2}{32l_2\left(Q+\frac{1}{3}ql_2\right)}-$$

$$-\frac{\left(\frac{1}{2}Qa+\frac{1}{3}qv_0^2+\frac{1}{3}qal_2\right)\left(Qa+\frac{1}{2}ql_2^2+\frac{1}{2}qal_2\right)}{8l_2^2\left(Q+\frac{1}{3}ql_2\right)^2},$$

$$\delta_{12}=\nu_2-\frac{\left[\alpha g-v_0\left(Q+\frac{1}{2}ql_2\right)^2\right]\nu_2^3}{8K^2g^2}-\frac{\left(Qa+\frac{1}{2}qv_0^2+\frac{1}{2}qal_2\right)\nu_2}{4Kg},$$

$$\delta_{22}=\frac{v_0\left(Q+\frac{2}{3}ql_2\right)\nu_2^3}{4Kg}-$$

$$-\frac{\left[\alpha g-v_0\left(Q+\frac{1}{2}ql_2\right)\right]\left[Qa+\frac{1}{2}(qv_0^2+qal_2)\right]\nu_2^3}{8K^2g^2},$$

$$\delta_{23}=\frac{v_0^2\left(Q+\frac{2}{3}ql_2\right)^2\nu_2^5}{8K^2g^2}-\frac{\left[\frac{1}{2}Qa+\frac{1}{3}q(v_0^2+al_2)\right]\nu_2^3}{6Kg}-$$

$$-\frac{\left[Qa+\frac{1}{2}q(v_0^2+al_2)\right]\nu_2^3}{24K^2g^2},$$

$$\delta_{24} = \frac{v_0 q a v_2^3}{24Kg} - \frac{3v_0\left(Q + \frac{2}{3}ql_2\right)\left[\frac{1}{2}Qa + \frac{1}{3}q(v_0^2 + al_2)\right]v_2^5}{16K^2g^2},$$

$$\delta_{25} = \frac{3\left|\frac{1}{2}Qa + \frac{1}{3}q(v_0^2 + al_2)\right|^2 v_2^5}{40K^2g^2} - \frac{qa^2v_2^3}{120Kg},$$

$$v_2 = \sqrt{\frac{Kg}{l_2\left(Q + \frac{1}{3}ql_2\right)}},$$

B_2, δ_{02} are constants of integration.

By using the resulting solutions we can compute the forces $T_1 = K_\varphi$ which occur in the thread when the load Q is raised (or lowered) for each region of loading or unloading. These calculations were carried out in reference 64 with the following initial data:

v_0=3.724 m/sec; a=0.98 m/sec^2; l_0=105 m;
Q= 5200 kg; q=2.48 kg/m; K= 4.45·10^{6}kg,
α=0 (figure 2).

Figure 3 shows the graph of forces in a visco-elastic thread when we have the following initial data:

v_0=3.6 m/sec; a=0.75 m/sec^2; l_0=182 m; Q=5200 kg,
q=2.6 kg/m; α=3000 kg·sec; K=4.64·10^6 kg.

The last example was also computed by the Sturm method. It was found that the discrepancy in results obtained by these two methods did not exceed 7 percent.

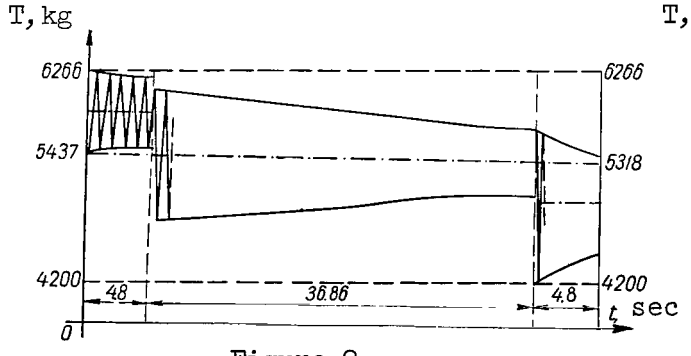

Figure 2

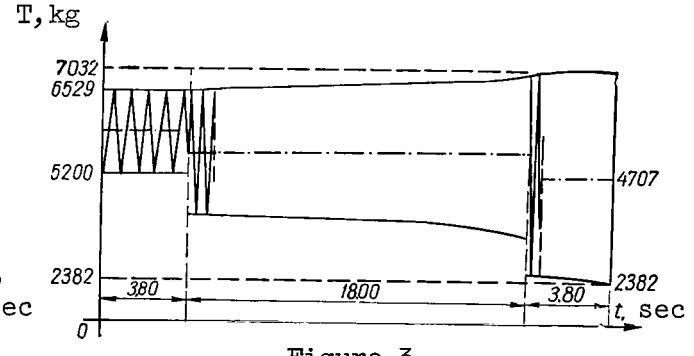

Figure 3

CHAPTER 2. CONSTRUCTION OF THE ASYMPTOTIC SOLUTION FOR A SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER WITH SLOWLY VARYING COEFFICIENTS

Section 6. Formulation of the Problem

Let us examine a system of linear differential equations of the following type in a finite-dimensional space: /31

$$A(\tau, \varepsilon)\frac{d^2x}{dt^2} + \varepsilon C(\tau, \varepsilon)\frac{dx}{dt} + B(\tau, \varepsilon)x = P(\tau, \varepsilon)e^{i\theta(t,\varepsilon)}, \tag{6.1}$$

where $A(\tau, \varepsilon)$, $C(\tau, \varepsilon)$, $B(\tau, \varepsilon)$ are square real matrices of order n while $x(t, \varepsilon)$, $P(\tau, \varepsilon)$ are n-dimensional vectors. It is proposed that the following formal expansions exist:

$$\begin{aligned} A(\tau, \varepsilon) &= \sum_{s=0}^{\infty} \varepsilon^s A_s(\tau), \quad C(\tau, \varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s C_s(\tau), \\ B(\tau, \varepsilon) &= \sum_{s=0}^{\infty} \varepsilon^s B_s(\tau), \quad P(\tau, \varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s P_s(\tau), \end{aligned} \tag{6.2}$$

in which $A_0(\tau), B_0(\tau)$ are symmetric, positively determined* matrices and the matrix $A_0(\tau)$ is nonsingular for all $\tau \in [0, L]$.

In particular, the sums (6.2) may also be finite.

In the present chapter (Sections 7-9) we shall present a method for constructing the asymptotic solution of the Cauchy problem for equation (6.1) with initial conditions

$$x(t, \varepsilon)|_{t=0} = x_0, \quad \left.\frac{dx}{dt}\right|_{t=0} = x_0 \tag{6.3}$$

and we shall indicate (Section 11) a method of finding the asymptotic eigenvalues of certain boundary value problems.

We shall now make several introductory remarks.

In providing the theorems of Chapter 2 it will be necessary for us to consider the solutions of a system of n linear algebraic equations

$$[B_0(\tau) - \omega(\tau)A_0(\tau)]\mu(\tau) = 0, \tag{6.4}$$

---

* In regard to this proposition see footnote on page 8.

where $\omega(\tau)=\omega_\nu(\tau)$ $(\nu=1, 2, \ldots, n)$ are the roots of the characteristic equation /32

$$\det[B_0(\tau)-\omega(\tau)A_0(\tau)]=0. \tag{6.5}$$

We shall assume that for any $\tau\in[0,L]$ all the roots of equation (6.5) are simple and are not equal to zero.

Furthermore, we normalize the mutually orthogonal solutions of system (6.4) $\mu_\nu(\tau)(\nu=1, 2, \ldots, n)$ in such a way that the following condition is satisfied:

$$(A_0(\tau)\mu_i(\tau),\mu_k(\tau))=\delta_{ik}, \tag{6.6}$$

where δ_{ik} is the Kronecker symbol.

Additionally, we let $\frac{d\theta(t,\varepsilon)}{dt}=k(\tau)$. Then we can have two cases depending on the values of function $k^2(\tau)$ and $\omega_\nu(\tau)$:

(1) The case of "resonance," when the function $k^2(\tau)$ becomes equal to one or several of the functions $\omega_\nu(\tau)$ at certain points of the segment $(0,L)$;

(2) The case of "nonresonance," when

$$k^2(\tau)\neq\omega_\nu(\tau), \quad \nu=1, 2, \ldots, n \tag{6.7}$$

for all $\tau\in[0,L]$.

Below we shall consider each case independently.

Section 7. Formal Solution in the Case of "Resonance"

We let

$$k^2(\tau)=\omega_1(\tau) \tag{7.1'}$$

for certain values of $\tau\in[0,L]$. However, $k^2(\tau)$ is not equal to $\omega_p(\tau)$ $(p=2, 3, \ldots, n)$ for any τ from segment $[0,L]$. In this case the following theorem is valid.

Theorem 2.1. If the matrices $A_s(\tau)$, $C_s(\tau)$, $B_s(\tau)$, the vectors $P_s(\tau)$ $(s=0, 1, 2, \ldots)$ and the function $k(\tau)$ contain derivatives of all orders in the interval $[0,L]$, then the formal particular solution of system (6.1) in the case of "resonance" may be presented in the form

$$x(t,\varepsilon)=[\Pi(\tau,\varepsilon)\xi+H(\tau,\varepsilon)]e^{i\theta(t,\varepsilon)}, \tag{7.1}$$

where the scalar function $\xi(t,\varepsilon)$ is determined by a first-order differential equation

$$\frac{d\xi}{dt} = [D(\tau,\ \varepsilon) + i(\Omega(\tau,\ \varepsilon) - k(\tau))]\,\xi + Z(\tau,\ \varepsilon), \tag{7.2}$$

while the vectors $\Pi(\tau,\varepsilon), H(\tau,\varepsilon)$ and the functions $D(\tau,\varepsilon), \Omega(\tau,\varepsilon), Z(\tau,\varepsilon)$ can be formally expanded in series of the form /33

$$\begin{gathered}\Pi(\tau,\ \varepsilon) = \mu_1(\tau) + \sum_{s=1}^{\infty} \varepsilon^s \Pi_s(\tau)^*,\\ H(\tau,\ \varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s H_s(\tau), \quad D(\tau,\ \varepsilon) = \sum_{s=1}^{\infty} \varepsilon^s D_s(\tau),\\ \Omega(\tau,\ \varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s \Omega_s(\tau), \quad Z(\tau,\ \varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s Z_s(\tau).\end{gathered} \tag{7.3}$$

Proof. We note first that the eigenvalues $\omega_\nu(\tau)$ and the eigenvectors $\mu_\nu(\tau)(\nu=1,\ 2,\ \ldots)$, according to reference 87, have the same number of derivatives with respect to τ on the segment $[0,L]$ as the matrices $A_0(\tau), B_0(\tau)$.

In order to determine the terms of the formal series (7.3) we substitute the vector $x(t,\varepsilon)$ from (7.1) into equation (6.1), while taking into account relationship (7.2). If in the resulting identities we equate separately the coefficients in front of the function $\xi(t,\varepsilon)$ and separately the free terms, we find two relationships:

$$\begin{gathered}A(\tau,\ \varepsilon)\{\varepsilon^2\Pi''(\tau,\ \varepsilon) + 2\varepsilon\Pi'(\tau,\ \varepsilon)[D(\tau,\ \varepsilon) + i\Omega(\tau,\ \varepsilon)] +\\ + \varepsilon\Pi(\tau,\ \varepsilon)[D(\tau,\ \varepsilon) + i\Omega(\tau,\ \varepsilon)]' + \Pi(\tau,\ \varepsilon)[D(\tau,\ \varepsilon) + i\Omega(\tau,\ \varepsilon)]^2\} +\\ + \varepsilon C(\tau,\ \varepsilon)\{\varepsilon\Pi'(\tau,\ \varepsilon) + \Pi(\tau,\ \varepsilon)[D(\tau,\ \varepsilon) + i\Omega(\tau,\ \varepsilon)]\} +\\ + B(\tau,\ \varepsilon)\Pi(\tau,\ \varepsilon) = 0,\end{gathered} \tag{7.4}$$

$$\begin{gathered}A(\tau,\varepsilon)\{[D(\tau,\ \varepsilon) + i(\Omega(\tau,\ \varepsilon) + k(\tau))]\Pi(\tau,\ \varepsilon)Z(\tau,\ \varepsilon) +\\ + \varepsilon Z'(\tau,\ \varepsilon)\Pi(\tau,\varepsilon) + 2\varepsilon Z(\tau,\ \varepsilon)\Pi'(\tau,\ \varepsilon) + i\varepsilon k'(\tau)H(\tau,\ \varepsilon) -\\ - k^2(\tau)H(\tau,\ \varepsilon) + 2i\varepsilon k(\tau)H'(\tau,\ \varepsilon) + \varepsilon^2 H''(\tau,\ \varepsilon)\} +\\ + \varepsilon C(\tau,\ \varepsilon)[\Pi(\tau,\ \varepsilon)Z(\tau,\ \varepsilon) + ik(\tau)H(\tau,\ \varepsilon) + \varepsilon H'(\tau,\ \varepsilon)] +\\ + B(\tau,\ \varepsilon)H(\tau,\ \varepsilon) = P(\tau,\ \varepsilon).\end{gathered} \tag{7.5}$$

---

* The case of "resonance" in which $k^2(\tau) = \omega_r(\tau)$ for certain $\tau \in [0,L]$, vector $\Pi(\tau,\varepsilon)$ has the form

$$\Pi(\tau,\varepsilon) = \mu_r(\tau) + \sum_{s=1}^{\infty} \varepsilon^s \Pi_s(\tau), \qquad 1 \leqslant r \leqslant n.$$

1. To determine the vectors $\Pi_s(\tau)$ of functions $D_{s+1}(\tau)$ and $\Omega_s(\tau)$ (s=0, 1, ...) we make use of relationship (7.4). By isolating the coefficients in front of the same powers of parameter ϵ in these, we obtain /34

$$[B_0(\tau) - \Omega_0^2(\tau)\ A_0(\tau)]\,\mu_1(\tau) = 0, \tag{7.6}$$

$$[B_0(\tau) - \Omega_0^2(\tau)\,A_0(\tau)]\,\Pi_s(\tau) = E_s(\tau),$$

$$s = 1,\ 2,\ \ldots, \tag{7.7}$$

where

$$E_s(\tau) = -A_{s-2}(\tau)\,\mu_1''(\tau) - \sum_{j=0}^{s-3} A_j(\tau)\,\Pi_{s-2-j}''(\tau) -$$

$$- 2\sum_{j=0}^{s-1} A_j(\tau)\,[D_{s-1-j}(\tau) + i\Omega_{s-1-j}(\tau)]\,\mu_1'(\tau) -$$

$$- 2\sum_{j_1=0}^{s-2}\sum_{j=1}^{s-j_1-1} A_{j_1}(\tau)\,\Pi_j'(\tau)\,[D_{s-j_1-j-1}(\tau) + i\Omega_{s-j_1-j-1}(\tau)] -$$

$$- \sum_{j=0}^{s-1} A_j(\tau)\,[D_{s-j-1}(\tau) + i\Omega_{s-j-1}(\tau)]'\,\mu_1(\tau) -$$

$$- \sum_{j_1=0}^{s-2}\sum_{j=1}^{s-j_1-1} A_{j_1}(\tau)\,\Pi_j(\tau)\,[D_{s-j_1-j-1}(\tau) + i\Omega_{s-j_1-j-1}(\tau)] -$$

$$- \sum_{j_2=1}^{s-1}\sum_{j_1=1}^{s-1-j_2}\sum_{j=1}^{s-1-j_1-j_2} A_{j_2}(\tau)\,\Pi_{j_1}(\tau)\,[D_j(\tau) + i\Omega_j(\tau)]\,[D_{s-j_2-j_1-j}(\tau) +$$

$$+ i\Omega_{s-j_2-j_1-j}(\tau)] - \sum_{j_1=0}^{s}\sum_{j=0}^{s-j_1} A_{j_1}(\tau)\,\mu_1(\tau)\,[D_j(\tau) + i\Omega_j(\tau)]\,[D_{s-j_1-j}(\tau) +$$

$$+ i\Omega_{s-j_1-j}(\tau)] - C_{s-2}(\tau)\,\mu_1'(\tau) - \sum_{j=0}^{s-3} C_j(\tau)\,\Pi_{s-2-j}'(\tau) -$$

$$- \sum_{j=0}^{s-1} C_j(\tau)\,[D_{s-1-j}(\tau) + i\Omega_{s-1-j}(\tau)] -$$

$$- \sum_{j_1=0}^{s-2}\sum_{j=1}^{s-1-j_1} C_{j_1}(\tau)\,\Pi_j(\tau)\,[D_{s-j_1-j-1}(\tau) + i\Omega_{s-j_1-j-1}(\tau)] -$$

$$- B_s(\tau)\,\mu_1(\tau) - \sum_{j=1}^{s-1} B_j(\tau)\,\Pi_{s-j}(\tau),$$

$$s = 1, 2, \ldots. \tag{7.8}$$

By taking the scalar product of relationships (7.6) and (6.4) (in the latter we assume that $\mu(\tau)=\mu_1(\tau)$) and the vector $\mu_1(\tau)$, and taking into account conditions (6.6), it is easy to find the function $\Omega_0(\tau)$: /35

$$\Omega_0(\tau) = \pm\sqrt{\omega_1(\tau)}.$$

In the future we assume that

$$\Omega_0(\tau) = +\sqrt{\omega_1(\tau)}. \tag{7.9}$$

Now let us proceed with the construction of functions $D_1(\tau)$,$\Omega_1(\tau)$ and of the vector $\Pi_1(\tau)$. For this purpose we make use of relationship (7.7) when s=1, specifically:

$$[B_0(\tau) - A_0(\tau)\omega_1(\tau)]\Pi_1(\tau) = E_1(\tau), \tag{7.10}$$

where

$$\begin{aligned} E_1(\tau) = {} & [A_1(\tau)\omega_1(\tau) - B_1(\tau)]\mu_1(\tau) - iA_0(\tau)[2\mu_1'(\tau)\Omega_0(\tau) - \\ & - \mu_1(\tau)\Omega_0'(\tau) + 2\Omega_0(\tau)(D_1(\tau) + i\Omega_1(\tau))\mu_1(\tau)] - \\ & - i\Omega_0(\tau)C_0(\tau)\mu_1(\tau). \end{aligned} \tag{7.11}$$

Thus the unknown vector $\Pi_1(\tau)$ is the solution of the nonhomogeneous system of linear algebraic equations (7.10) whose right side--the vector $E_1(\tau)$--in turn depends on the unknown function $D_1(\tau)+i\Omega_1(\tau)$. We also note that the determinant $\det|B_0(\tau)-\omega_1(\tau)A_0(\tau)|$ of system (7.10), in accordance with the proposition adopted earlier (see (6.5)), becomes equal to zero at the points of the segment [0,L]. In view of this we can find the vector $\Pi_1(\tau)$ and the function $D_i(\tau)+i\Omega_1(\tau)$ by making use of the well-known propositions of linear algebra (ref. 68): a nonhomogeneous system of linear algebraic equations

$$TX=f,$$

whose determinant is equal to zero, can be solved only when the vector f is orthogonal with respect to all vectors which give the solution of the homogeneous system $T^*y=0$. (T^* is a matrix which is the conjugate of T.)

In our case the matrices $B_0(\tau)$ and $A_0(\tau)$ are symmetric and all roots of equation (6.5) are simple. Therefore for the points where the determinant of system (7.10) is annulled, the necessary and sufficient condition for the solution of the system is as follows:

$$(E_s(\tau),\ \mu_1(\tau)) = 0, \qquad s = 1,\ 2,\ \ldots. \tag{7.12}$$

We make use of this condition to determine the function $D_1(\tau)+i\Omega_1(\tau)$ on the entire segment $[0,L]$. Then, according to (7.11), we find

$$D_1(\tau) + i\Omega_1(\tau) = \frac{1}{2i\Omega_0(\tau)}[(A_1(\tau)\mu_1(\tau),\ \mu_1(\tau))\,\omega_1(\tau) -$$
$$-(B_1(\tau)\mu_1(\tau),\ \mu_1(\tau)) - 2i\Omega_0(\tau)(A_0(\tau)\mu_1'(\tau),\ \mu_1(\tau)) -$$
$$-i\Omega_0'(\tau) - i(C_0(\tau)\mu_1(\tau),\ \mu_1(\tau))\,\Omega_0(\tau)],$$

or

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$$D_1(\tau) = -(A_0(\tau)\mu_1'(\tau),\ \mu_1(\tau)) - \frac{\Omega_0'(\tau)}{2\Omega_0(\tau)} -$$
$$-\frac{1}{2}(C_0(\tau)\mu_1(\tau),\ \mu_1(\tau)), \tag{7.13}$$
$$\Omega_1(\tau) = \frac{(A_1(\tau)\mu_1(\tau),\ \mu_1(\tau))\,\omega_1(\tau) - (B_1(\tau)\mu_1(\tau),\ \mu_1(\tau))}{2\Omega_0}.$$

Having determined the function $D_1(\tau)+i\Omega_1(\tau)$ and consequently the vector $E_1(\tau)$, we can find the unknown vector $\Pi_1(\tau)$.

We shall seek $\Pi_1(\tau)$ in the form

$$\Pi_1(\tau) = \sum_{j=1}^{n} V_{j_1}(\tau)\mu_j(\tau), \tag{7.14}$$

where $V_{j_1}(\tau)$ are scalar functions which must be determined.

Substituting (7.14) into relationship (7.10) and taking the scalar product of the result and the vector $\mu_r(\tau)$, we find

$$[\omega_r(\tau) - \omega_1(\tau)]\,V_{r_1}(\tau) = (E_1(\tau),\ \mu_r(\tau)). \tag{7.15}$$

From this it follows that

$$V_{p_1}(\tau) = \frac{(E_1(\tau),\ \mu_p(\tau))}{\omega_p(\tau) - \omega_1(\tau)}, \qquad p = 2,\ 3,\ \ldots,\ n. \tag{7.16}$$

In view of equalities (7.12) and (7.15), the function $V_{11}(\tau)$ remains arbitrary. We set it equal to zero.

In the same way we can determine all the subsequent functions $D_s(\tau), \Omega_s(\tau)$ and the vectors $\Pi_s(\tau)$ (s=2, 3, ...). Indeed let us assume that the specified method has been used to find all the vectors $\Pi_s(\tau)$ and the functions $D_s(\tau)$, $\Omega_s(\tau)$, whose numbers are less than some number m. Then, by making use of relationship (7.7) with s=m, we can determine the functions $D_m(\tau), \Omega_m(\tau)$ and the vector $\Pi_m(\tau)$ from the system of equations

$$[B_0(\tau) - \omega_1(\tau) A_0(\tau)] \Pi_m(\tau) = E_m(\tau). \tag{7.17}$$

As in the case s=1, we first find the function $D_m(\tau)+i\Omega_m(\tau)$ from the condition

$$(E_m(\tau),\ \mu_1(\tau)) = 0, \tag{7.18}$$

and then we construct the vector $\Pi_m(\tau)$ in the form of a sum /37

$$\Pi_m(\tau) = \sum_{j=1}^{n} V_{jm}(\tau)\mu_j(\tau), \tag{7.19}$$

where the scalar functions $V_{jm}(\tau)$ are given by the relationships

$$V_{jm} = \frac{(E_m(\tau),\ \mu_j(\tau))}{\omega_j(\tau) - \omega_1(\tau)}, \qquad j = 2,\ 3,\ \ldots,\ n. \tag{7.20}$$

As before, the arbitrary function V_{1m} is set equal to zero.

Thus our algorithm makes it possible to find the functions $D_m(\tau), \Omega_m(\tau)$ and the vector $\Pi_m(\tau)$ for any natural m.

It follows from the form of the resulting equations (from the conditions of the theorem) that any number of derivatives with respect to τ exists for functions $D_s(\tau), \Omega_s(\tau)$ and vectors $\Pi_s(\tau)$ (s=0, 1, ..., m) on the segment [0,L].

2. Let us now find the vector $H(\tau,\varepsilon)$ and the function $Z(\tau,\varepsilon)$ in relationships (7.1) and (7.2). To do this we compare the coefficients in front of the same powers of parameter ε in identity (7.5). As a result we obtain

$$[B_0(\tau) - k^2(\tau) A_0(\tau)] H_s(\tau) = F_s(\tau), \quad s = 0, 1, \ldots, \tag{7.21}$$

where

$$\begin{aligned}
F_s(\tau) = P_s(\tau) &- ik(\tau) \sum_{j=0}^{s} A_j(\tau) \mu_1(\tau) Z_{s-j}(\tau) - \\
&- \sum_{j=0}^{s} \sum_{j_1=0}^{s-j} A_j(\tau) \mu_1(\tau) (D_{j_1}(\tau) + i\Omega_{j_1}(\tau)) Z_{s-j-j_1}(\tau) - \\
&- \sum_{j=0}^{s} \sum_{j_1=0}^{s-j} \sum_{j_2=0}^{s-j-j_1} A_j(\tau) \Pi_{j_1}(\tau) (D_{j_2}(\tau) + i\Omega_{j_2}(\tau)) Z_{s-j_1-j_2-j}(\tau) - \\
&- \sum_{j=0}^{s-1} A_j(\tau) \mu_1(\tau) Z'_{s-1-j}(\tau) - \sum_{j=0}^{s-1} \sum_{j_1=1}^{s-1-j} A_j(\tau) \Pi_{j_1}(\tau) Z'_{s-1-j-j_1}(\tau) - \\
&- \sum_{j=0}^{s-1} A_j(\tau) \mu_1(\tau) Z'_{s-1-j}(\tau) - \sum_{j=0}^{s-1} \sum_{j_1=1}^{s-1-j} A_j(\tau) \Pi_{j_1}(\tau) Z'_{s-1-j_1-j}(\tau) - \\
&- 2 \sum_{j=0}^{s-1} A_j(\tau) \mu'_1(\tau) Z_{s-1-j}(\tau) - \sum_{j=0}^{s-1} \sum_{j_1=1}^{s-1-j} A_j(\tau) \Pi'_{j_1}(\tau) Z_{s-1-j_1-j}(\tau) - \\
&- ik'(\tau) \sum_{j=0}^{s-1} A_j(\tau) H_{s-1-j}(\tau) + k^2(\tau) \sum_{j=1}^{s} A_j(\tau) H_{s-j}(\tau) - \\
&- 2ik(\tau) \sum_{j=0}^{s-1} A_j(\tau) H'_{s-1-j}(\tau) - \sum_{j=0}^{s-2} A_j(\tau) H'_{s-2-j}(\tau) - \\
&- \sum_{j=0}^{s-1} C_j(\tau) \mu_1(\tau) Z_{s-1-j}(\tau) - \sum_{j=0}^{s-2} \sum_{j_1=1}^{s-1-j} C_j(\tau) \Pi_{j_1}(\tau) Z_{s-j_1-j-1}(\tau) - \\
&- ik(\tau) \sum_{j=0}^{s-1} C_j(\tau) H_{s-1-j}(\tau) - \sum_{j=0}^{s-2} C_j(\tau) H_{s-2-j}(\tau) - \\
&- \sum_{j=1}^{s} B_j(\tau) H_{s-j}(\tau), \quad s = 0, 1, 2, \ldots .
\end{aligned} \tag{7.22}$$

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Thus the determination of vectors $H_s(\tau)$ and of functions $Z_s(\tau)$ ($s=0, 1, 2, \ldots$) is reduced to the solution of a system of algebraic equations (7.21), which is analogous to system (7.7), investigated by us in Section 1. We recall that the determinant of system (7.21) for certain values of τ from $[0,L]$ becomes equal to zero because we are considering the construction of a formal solution in the case of "resonance," i.e., for certain values of τ the terms $k^2(\tau)$ and $\omega_1(\tau)$ coincide (see (7.1')).

In view of the above we shall find the unknown function $z_s(\tau)$(s=0, 1, 2, ..) from the condition

$$(F_s(\tau),\ \mu_1(\tau)) = 0, \qquad s = 0,\ 1,\ \ldots. \tag{7.23}$$

After this it is easy to construct the vector $H_s(\tau)$ if we seek it as a solution of system (7.21) in the form

$$H_s(\tau) = \sum_{j=1}^{n} C_{js}(\tau)\mu_j(\tau), \qquad s = 0,\ 1,\ 2,\ \ldots, \tag{7.24}$$

where $C_{js}(\tau)$ are the coefficients which must be determined.

Indeed, by substituting (7.24) into (7.21) and taking the scalar product of the result and $\mu_r(\tau)$, we find the unknown coefficients $C_{js}(\tau)$:

$$C_{rs}(\tau) = \frac{(F_s(\tau),\ \mu_r(\tau))}{\omega_r(\tau) - k^2(\tau)}, \qquad r = 2,\ 3,\ \ldots,\ n. \tag{7.25}$$

According to equalities (7.23) and (7.1'), the function $C_{js}(\tau)$(s=0, ...,) remains arbitrary. We set it equal to zero. Thus the specified method may be used to determine the values of $Z_s(\tau)$ and $H_s(\tau)$ successively for any natural s.

For example, when s=0, we have /39

$$Z_0(\tau) = -i\frac{(P_0(\tau),\ \mu_1(\tau))}{k(\tau) + \Omega_0(\tau)},$$

and when s=1

$$\begin{aligned} Z_1(\tau) = -\frac{i}{\Omega_0(\tau) + k(\tau)}\{&(P_1(\tau),\ \mu_1(\tau)) - ik(\tau)(A_1(\tau)\mu_1(\tau),\ \mu_1(\tau)) - \\ &- [D_1(\tau) + i\Omega_1(\tau)]Z_0(\tau) - i(A_1(\tau)\mu_1(\tau),\ \mu_1(\tau))\Omega_0(\tau)Z_0(\tau) - \\ &- i(A_0(\tau)\Pi_1(\tau),\ \mu_1(\tau))\Omega_0(\tau)Z_0(\tau) - Z_0'(\tau) - \\ &- 2(A_0(\tau)\mu_1'(\tau),\ \mu_1(\tau)) + k^2(\tau)(A_1(\tau)H_0(\tau),\ \mu_1(\tau)) - \\ &- ik'(\tau)(A_0(\tau)H_1(\tau),\ \mu_1(\tau)) - 2ik(\tau)(A_0(\tau)H_0(\tau),\ \mu_1(\tau)) - \\ &- (C_0(\tau)\mu_1(\tau),\ \mu_1(\tau))Z_0(\tau) - ik(\tau)(C_0(\tau)H_0(\tau),\ \mu_1(\tau)) - \\ &- (B_0(\tau)H_0(\tau),\ \mu_1(\tau))\}. \end{aligned} \tag{7.26}$$

It is easy to write out the corresponding equations which are used to determine the coefficients $C_{rs}(\tau)$ ($r=2, 3, \ldots, n$; $s=0, 1, 2, \ldots$).

Thus Theorem 2.1 has been proved.

Note 1. Theorem 2.1 may be extended to the more general case of "resonance," specifically: 1) when the value of the function $k^2(\tau)$ for individual $\tau\in[0,L]$ coincides with certain roots $\omega_r(\tau)$ ($1\leq r\leq n$) of equation (6.5); 2) when the right side of equation (6.1) has the form $\sum_{n=1}^{N} P_n(\tau,\varepsilon)e^{i\theta n}$ and the values of various functions $k_n^2(\tau)$ coincide with various roots of equation (6.5).

Note 2. To construct the formal general solution of system (6.1) we should add to the resulting particular solution (7.1) the formal general solution of the corresponding homogeneous system

$$A(\tau,\ \varepsilon)\frac{d^2x}{dt^2}+\varepsilon C(\tau,\ \varepsilon)\frac{dx}{dt}+B(\tau,\ \varepsilon)x=0. \tag{7.27}$$

A certain ν-th ($\nu=1, 2, \ldots, n$) particular solution of system (7.27) is sought with the aid of the above algorithm in the form

$$x_\nu(t,\ \varepsilon)=\Pi_\nu(\tau,\ \varepsilon)\xi_\nu(t,\ \varepsilon), \tag{7.28}$$

where $\Pi_\nu(\tau,\varepsilon)$ is an n-dimensional vector of form (7.3) while $\xi_\nu(\tau,\varepsilon)$ is a scalar function which satisfies the first-order differential equation

$$\frac{d\xi_\nu}{dt}=[D_\nu(\tau,\ \varepsilon)+i\Omega_\nu(\tau,\ \varepsilon)]\,\xi_\nu, \tag{7.29}$$
$$\nu=1,\ 2,\ \ldots,\ n,$$

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in which

$$D_\nu(\tau,\ \varepsilon)=\sum_{s=1}^{\infty}\varepsilon^s D_{\nu s}(\tau),\quad \Omega_\nu(\tau,\ \varepsilon)=\sum_{s=0}^{\infty}\varepsilon^s\Omega_{\nu s}(\tau), \tag{7.30}$$
$$\nu=1,\ 2,\ \ldots,\ n.$$

Here the functions $D_{\nu s}(\tau)$, $\Omega_{\nu s}(\tau)$ and the vectors $\Pi_{\nu s}(\tau)$ ($s=0, 1, \ldots$; $\nu=1, 2, \ldots, n$) are determined from the corresponding equations of this section in which we must replace $\omega_1(\tau)$, $\mu_1(\tau)$ by $\omega_\nu(\tau)$, $\mu_\nu(\tau)$. Since all the $\omega_\nu(\tau)$ ($\nu=1, 2, \ldots, n$) are different, we can use both values of the function $\Omega_{\nu s}(\tau)$ and obtain

2n particular linearly independent solutions which are necessary for constructing the formal general solution of equation (7.27).

Section 8. The Case of "Nonresonance"

The following theorem can be proved for the case of "nonresonance."

Theorem 2.2. If the coefficients of equation (6.1) and the function $k(\tau)$ on the segment $[0, L]$ have derivatives of all orders with respect to τ, the formal general solution of system (6.1) in the case of "nonresonance" may be represented in the form

$$x(t, \varepsilon) = \sum_{\nu=1}^{n} \Pi_\nu(\tau, \varepsilon)\, \xi_\nu(t, \varepsilon) + R(\tau, \varepsilon)\, e^{i\theta(t,\varepsilon)}, \tag{8.1}$$

where $\sum_{\nu=1}^{n} \Pi_\nu(\tau, \varepsilon)\, \xi_\nu(t, \varepsilon)$ is the general solution of equation (7.27) while $R(\tau, \varepsilon)$ is an n-dimensional vector which can be expanded in the form

$$R(\tau, \varepsilon) = \sum_{s=0}^{n} \varepsilon^s R_s(\tau). \tag{8.2}$$

Proof. In view of Note 2 in Section 7, the proof of this theorem is reduced to the determination of the elements of the formal series (8.2). For this purpose we substitute

$$\tilde{x}(t, \varepsilon) = R(\tau, \varepsilon)\, e^{i\theta(t,\varepsilon)} \tag{8.3}$$

into system (6.1) and in the resulting identity

$$\begin{aligned} &A(\tau, \varepsilon)\{\varepsilon^2 R''(\tau, \varepsilon) + \varepsilon i[2k(\tau) R'(\tau, \varepsilon) + k'(\tau) R(\tau, \varepsilon)] - \\ &- k^2(\tau) R(\tau, \varepsilon)\} + \varepsilon^2 C(\tau, \varepsilon)[R'(\tau, \varepsilon) + ik'(\tau) R(\tau, \varepsilon)] + \\ &+ B(\tau, \varepsilon) R(\tau, \varepsilon) = P(\tau, \varepsilon) \end{aligned} \tag{8.4}$$

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we compare the coefficients of ε^s ($s=0, 1, \ldots$). As a result we find the following recurrent relationships:

$$[B_0(\tau) - k^2(\tau) A_0(\tau)] R_s(\tau) = G_s(\tau), \qquad s = 0, 1, \ldots, \tag{8.5}$$

where

$$\begin{aligned} G_s(\tau) = P_s(\tau) &- \sum_{j=0}^{s-2} A_j(\tau) R_{s-2-j}(\tau) - 2ik(\tau) \sum_{j=0}^{s-1} A_j(\tau) R'_{s-1-j}(\tau) - \\ &- ik'(\tau) \sum_{j=0}^{s-1} A_j(\tau) R_{s-1-j}(\tau) - \sum_{j=0}^{s-2} C_j(\tau) R'_{s-2-j}(\tau) - \end{aligned}$$

$$-ik(\tau)\sum_{j=0}^{s-1}C_j(\tau)R_{s-1-j}(\tau)-\sum_{j=1}^{s}B_j(\tau)R_{s-j}(\tau). \tag{8.6}$$

Since according to (6.7)

$$\det[B_0(\tau)-k^2(\tau)A_0(\tau)]\neq 0 \tag{8.7}$$

for any $\tau\in[0,L]$, we use relationship (8.5) to find

$$R_s(\tau)=[B_0(\tau)-k^2(\tau)A_0(\tau)]^{-1}G_s(\tau),\qquad s=0,\ 1,\ \ldots. \tag{8.8}$$

We note that in this case the vector $G_s(\tau)$, during the successive solution of system (8.5), is determined in terms of known quantities.

The theorem has been proved.

Section 9. The Asymptotic Nature of the Solution

As was already noted in the Introduction, the practical applicability of the asymptotic method which is discussed here is determined not by the convergence of series (7.3) or (8.2) (in many cases these series are divergent), but rather by the asymptotic behavior of the vector $x^{(m)}(t,\varepsilon)$, which is determined in the following manner:

$$\begin{aligned} x^{(m)}(t,\ \varepsilon)&=[\Pi^{(m)}(\tau,\ \varepsilon)\xi^{(m)}(t,\ \varepsilon)+H^{(m)}(\tau,\ \varepsilon)]\,e^{i\theta(t,\varepsilon)},\\ \frac{d\xi^{(m)}}{dt}&=[D^{(m)}(\tau,\ \varepsilon)+i(\Omega^{(m)}(\tau,\ \varepsilon)-k(\tau))]\,\xi^{(m)}+Z^{(m)}(\tau,\ \varepsilon), \end{aligned} \tag{9.1}$$

where

$$\Pi^{(m)}(\tau,\ \varepsilon)=\mu_1(\tau)+\sum_{s=1}^{m}\varepsilon^s\Pi_s(\tau),\quad H^{(m)}(\tau,\ \varepsilon)=\sum_{s=0}^{m}\varepsilon^sH_s(\tau), \tag{9.2}$$

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$$\begin{aligned} D^{(m)}(\tau,\ \varepsilon)&=\sum_{s=1}^{m}\varepsilon^sD_s(\tau),\quad \Omega^{(m)}(\tau,\ \varepsilon)=\sum_{s=0}^{m}\varepsilon^s\Omega_s(\tau),\\ Z^{(m)}(\tau,\ \varepsilon)&=\sum_{s=0}^{m}\varepsilon^sZ_s(\tau) \end{aligned} \tag{9.3}$$

(m is any natural number).

Below it will be shown that the norm of the difference between the exact particular solution $x(t,\varepsilon)$ of system (6.1) and the corresponding vector

$x^{(m)}(t, \varepsilon)$ of the m-th approximation may be made arbitrarily small when ε is sufficiently small.

For this purpose we shall consider, for example, the case of "resonance" (analogous results may also be obtained for the case of "nonresonance").

We shall first prove the following lemma.

Lemma 2.1. If the conditions of Theorem 2.1 are satisfied, the m-th approximation satisfies the initial system of differential equations (6.1) with an accuracy which is determined by the infinitesimal of order ε^m uniformly with respect to t on the segment $0 \leq t \leq \frac{L}{\varepsilon}$ for any ε in the interval $0 < \varepsilon \leq \varepsilon_0$.

Proof. Indeed, if we substitute the vector $x^{(m)}(t, \varepsilon)$ determined by equation (9.1) into the expression

$$A(\tau, \varepsilon)\frac{d^2x^{(m)}}{dt^2} + \varepsilon C(\tau, \varepsilon)\frac{dx^{(m)}}{dt} + B(\tau, \varepsilon)x^{(m)} - P(\tau, \varepsilon)e^{i\theta(t,\varepsilon)},$$

we obtain

$$\begin{aligned}
&A(\tau, \varepsilon)\frac{d^2x^{(m)}}{dt^2} + \varepsilon C(\tau, \varepsilon)\frac{dx^{(m)}}{dt} + B(\tau, \varepsilon)x^{(m)} - P(\tau, \varepsilon)e^{i\theta(t,\varepsilon)} = \\
&= \{A(\tau, \varepsilon)[\varepsilon^2\Pi^{(m)''}(\tau, \varepsilon) + 2\varepsilon\Pi^{(m)'}(\tau, \varepsilon)(D^{(m)}(\tau, \varepsilon) + i\Omega^{(m)}(\tau, \varepsilon)) + \\
&+ \varepsilon\Pi^{(m)}(\tau, \varepsilon)(D^{(m)}(\tau, \varepsilon) + i\Omega^{(m)}(\tau, \varepsilon))' + \Pi^{(m)}(\tau, \varepsilon)(D^{(m)}(\tau, \varepsilon) + \\
&+ i\Omega(\tau, \varepsilon))^2] + \varepsilon C(\tau, \varepsilon)[\varepsilon\Pi^{(m)'}(\tau, \varepsilon) + \Pi^{(m)}(\tau, \varepsilon)(D^{(m)}(\tau, \varepsilon) + \\
&+ i\Omega^{(m)}(\tau, \varepsilon))] + B(\tau, \varepsilon)\Pi^{(m)}(\tau, \varepsilon)\}\xi^{(m)}(t, \varepsilon)e^{i\theta(t,\varepsilon)} + \qquad (9.4) \\
&+ \{A(\tau, \varepsilon)[(D^{(m)}(\tau, \varepsilon) + i\Omega^{(m)}(\tau, \varepsilon) - ik(\tau))\Pi^{(m)}(\tau, \varepsilon)Z^{(m)}(\tau, \varepsilon) + \\
&+ \varepsilon Z^{(m)'}(\tau, \varepsilon)\Pi^{(m)}(\tau, \varepsilon) + 2\varepsilon Z^{(m)}(\tau, \varepsilon)\Pi^{(m)'}(\tau, \varepsilon) + \\
&+ \varepsilon ik'(\tau)H^{(m)}(\tau, \varepsilon) - k^2(\tau)H^{(m)}(\tau, \varepsilon) + 2\varepsilon ik(\tau)H^{(m)'}(\tau, \varepsilon) + \\
&+ \varepsilon^2 H^{(m)''}(\tau, \varepsilon)] + \varepsilon C(\tau, \varepsilon)[\Pi^{(m)}(\tau, \varepsilon)Z^{(m)}(\tau, \varepsilon) + \\
&+ ik(\tau)H^{(m)}(\tau, \varepsilon) + \varepsilon H^{(m)'}(\tau, \varepsilon)] + B(\tau, \varepsilon)H^{(m)}(\tau, \varepsilon) - \\
&- P(\tau, \varepsilon)\}e^{i\theta(t,\varepsilon)}
\end{aligned}$$

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From the determination of elements in series (7.3), it follows that expressions in the right side of relationship (9.4) are infinitesimals of order ε^{m+1}. Therefore expression (9.4) may be represented in the form

$$A(\tau,\ \varepsilon)\frac{d^2x^{(m)}}{dt^2}+\varepsilon C(\tau,\ \varepsilon)\frac{dx^{(m)}}{dt}+B(\tau,\ \varepsilon)x^{(m)}= \tag{9.5}$$
$$=P(\tau,\ \varepsilon)e^{i\theta(t,\varepsilon)}+\varepsilon^{m+1}[F_m(\tau,\ \varepsilon)\xi^{(m)}(t,\ \varepsilon)+R_m(\tau,\ \varepsilon)]e^{i\theta(t,\varepsilon)},$$

where $F_m(\tau,\varepsilon),R_m(\tau,\varepsilon)$ are vectors which are regular with respect to ε in the neighborhood of the point $\varepsilon=0$.

Integrating equation (9.2), we find

$$\xi^{(m)}(t,\ \varepsilon)=ae^{\int_0^t D^{(m)}(\tau,\varepsilon)dt}\left\{\cos\int_0^t[\Omega^{(m)}(\tau,\ \varepsilon)-k(\tau)]\,dt+\right.$$
$$\left.+i\sin\int_0^t[\Omega^{(m)}(\tau,\ \varepsilon)-k(\tau)]\,dt\right\}+ \tag{9.6}$$
$$+\int_0^t Z^{(m)}(\tau_1,\ \varepsilon)e^{\int_{t_1}^t D^{(m)}(\tau,\varepsilon)dt}\left[\cos\int_{t_1}^t(\Omega^{(m)}(\tau,\ \varepsilon)-k(\tau))\,dt+\right.$$
$$\left.+i\sin\int_{t_1}^t(\Omega^{(m)}(\tau,\ \varepsilon)-k(\tau))\,dt\right]dt_1,$$

where a is the constant of integration.

Since the functions $D^{(m)}(\tau,\varepsilon),Z^{(m)}(\tau,\varepsilon),\Omega^{(m)}(\tau,\varepsilon),k(\tau)$ can be differentiated with respect to τ and are therefore bounded on the segment $[0,L]$, we can obtain, from relationship (9.6),

$$|\xi^{(m)}(t,\ \varepsilon)|\leqslant\frac{C_1}{\varepsilon}, \tag{9.7}$$

where C_1 is a constant which does not depend on ε.

Taking into account estimate (9.7), relationship (9.5) may be rewritten ir the form

$$A(\tau,\ \varepsilon)\frac{d^2x^{(m)}}{dt^2}+\varepsilon C(\tau,\ \varepsilon)\frac{dx^{(m)}}{dt}+B(\tau,\ \varepsilon)x^{(m)}= \tag{9.8}$$
$$=P(\tau,\ \varepsilon)e^{i\theta(t,\varepsilon)}+\varepsilon^m\Phi_m(\tau,\ \varepsilon),$$

where $\Phi_m(\tau,\varepsilon)$ is a vector which is uniformly bounded for all $t\in[0,\frac{L}{\varepsilon}]$ and /44

$0<\varepsilon\leq\varepsilon_0$.

The lemma has been proved.

Now we can prove the theorem which establishes the asymptotic nature of solution $x^{(m)}(t,\varepsilon)$.

Theorem 2.3. Let the exact solution $x(t,\varepsilon)$ of system (6.1) and the m-th approximation $x^{(m)}(t,\varepsilon)$ be taken under the same initial conditions

$$x(0) = x^{(m)}(0) = x_0, \quad \left.\frac{dx}{dt}\right|_{t=0} = \left.\frac{dx^{(m)}}{dt}\right|_{t=0} = x_0. \tag{9.9}$$

Then, when the conditions of Theorem 2.1 are satisfied, for every $L>0$ there is a constant C which is independent of ε such that for all $\tau\in[0,\frac{L}{\varepsilon}]$ and $0<\varepsilon\leq\varepsilon_0$ the following inequalities will be satisfied

$$\begin{aligned} \| x(t,\varepsilon) - x^{(m)}(t,\varepsilon) \| &\leqslant \varepsilon^{m-1} C, \\ \left\| \frac{dx}{dt} - \frac{dx^{(m)}}{dt} \right\| &\leqslant \varepsilon^{m-1} C. \end{aligned} \tag{9.10}$$

Proof. To prove this theorem we introduce the vector

$$y(t,\varepsilon) = x(t,\varepsilon) - x^{(m)}(t,\varepsilon). \tag{9.11}$$

It is easy to see that the vector $y(t,\varepsilon)$ satisfies the zero initial conditions

$$y(0) = 0, \qquad \left.\frac{dy}{dt}\right|_{t=0} = 0$$

and the system

$$A(\tau,\varepsilon)\frac{d^2y}{dt^2} + \varepsilon C(\tau,\varepsilon)\frac{dy}{dt} + B(\tau,\varepsilon)y = -\varepsilon^m \Phi_m(\tau,\varepsilon). \tag{9.12}$$

We can rewrite the resulting system (9.12) in the following manner:

$$\begin{aligned} A_0(\tau)\frac{d^2y}{dt^2} + B_0(\tau)y = &-\varepsilon A_1(\tau,\varepsilon)\frac{d^2y}{dt^2} - \varepsilon C(\tau,\varepsilon)\frac{dy}{dt} - \\ &- \varepsilon B_1(\tau,\varepsilon)y - \varepsilon^m \Phi_m(\tau,\varepsilon), \end{aligned} \tag{9.13}$$

where

$$A_1(\tau,\varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s A_{s+1}(\tau), \quad B_1(\tau,\varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s B_{s+1}(\tau). \tag{9.14}$$

We represent the unknown vector $y(t, \varepsilon)$ in the form /45

$$y(t, \varepsilon) = \sum_{j=1}^{n} \mu_j(\tau)\, b_j(t, \varepsilon), \tag{9.15}$$

where $b_j(t, \varepsilon)$ are scalar functions.

Substituting (9.15) into system (9.13) and taking the scalar product of the result and of the vector μ_k ($k=1, 2, \ldots, n$), we obtain

$$\begin{aligned} \frac{d^2 b_k}{dt^2} + \varrho_k^2(\tau)\, b_k = -\varepsilon \sum_{j=1}^{n} \{ & \varepsilon^2 (A_1(\tau, \varepsilon)\, \mu_j''(\tau), \mu_k(\tau)) + \\ & + \varepsilon (C(\tau, \varepsilon)\, \mu_j'(\tau), \mu_k(\tau)) + (B_1(\tau, \varepsilon)\, \mu_j(\tau), \mu_k(\tau))\, b_j + \\ & + (A_1(\tau, \varepsilon)\, \mu_j(\tau), \mu_k(\tau))\, \frac{d^2 b_j}{dt^2} + (C(\tau, \varepsilon)\, \mu_j(\tau), \mu_k(\tau))\, \frac{db_j}{dt} - \\ & - \varepsilon^m (\Phi_m(\tau, \varepsilon), \mu_k(\tau)), \qquad k = 1, 2, \ldots, n, \end{aligned} \tag{9.16}$$

where

$$\varrho_k^2(\tau) = \omega_k(\tau), \qquad k = 1, 2, \ldots, n. \tag{9.17}$$

We solve the resulting system (9.16) with respect to the second derivatives, noting that the determinant of the system, for sufficiently small values of ε, differs little from unity. We shall write the result in the form

$$\begin{aligned} \frac{d^2 b_k}{dt^2} + \varrho_k^2(\tau, \varepsilon)\, b_k = \varepsilon \sum_{j=1}^{n} \left[\alpha_{kj}(\tau, \varepsilon)\, b_j + \beta_{kj}(\tau, \varepsilon)\, \frac{db_j}{dt} \right] + \\ + \varepsilon^m f_k(\tau, \varepsilon), \qquad k = 1, 2, \ldots, n, \end{aligned} \tag{9.18}$$

where the functions $\varrho_k^2(\tau, \varepsilon)$, $\alpha_{kj}(\tau, \varepsilon)$, $\beta_{kj}(\tau, \varepsilon)$ and $f_k(\tau, \varepsilon)$ are bounded on the segment $[0, L]$, and according to (9.17),

$$\varrho_k^2(\tau, \varepsilon) \neq 0 \tag{9.19}$$

for any $\tau \in [0, L]$, $0 < \varepsilon \leq \varepsilon_0$.

Applying Lemma 1.2 to system (9.18), we have

$$|b_k(t, \varepsilon)| \leqslant S_k \int_0^t \left| \frac{P_k(\tau, \varepsilon)}{\varrho_k(\tau, \varepsilon)} \right| dt, \tag{9.20}$$

$$\left|\frac{db_k}{dt}\right| \leqslant S_{1k}\int_0^t \left|\frac{P_k(\tau,\varepsilon)}{\varrho_k(\tau,\varepsilon)}\right| dt,$$

where

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$$P_k(\tau,\varepsilon) = \varepsilon\sum_{j=1}^{n}\left[\alpha_{kj}(\tau,\varepsilon)\,b_j(t,\varepsilon) + \beta_{kj}(\tau,\varepsilon)\,\frac{db_j}{dt}\right] + \varepsilon^m f_k(\tau,\varepsilon). \tag{9.21}$$

We let

$$W = \sum_{j=1}^{n}\left(|b_j(t,\varepsilon)| + \left|\frac{db_j}{dt}\right|\right). \tag{9.22}$$

Adding inequalities (9.20) term by term, and noting that the functions $\alpha_{kj}(\tau,\varepsilon)$, $\beta_{kj}(\tau,\varepsilon)$, $\varrho_k(\tau,\varepsilon)$ and $f_k(\tau,\varepsilon)$ are bounded for any $\tau\in[0,L]$ and $0<\varepsilon\leq\varepsilon_0$, we find

$$W \leqslant \varepsilon K\int_0^t W dt + \varepsilon^{m-1}K_1, \tag{9.23}$$

where K, K_1 are constants which are independent of ε.

Applying Lemma 1.1 to inequality (9.23), we obtain the inequality

$$W \leqslant \varepsilon^{m-1}C, \quad C = K_1 e^{KL}. \tag{9.24}$$

From this we obtain the estimate

$$|b_j(t,\varepsilon)| \leqslant \varepsilon^{m-1}C, \quad \left|\frac{db_j}{dt}\right| \leqslant \varepsilon^{m-1}C, \quad j = 1, 2, \ldots, n. \tag{9.25}$$

The validity of Theorem 2.3 follows from inequality (9.25) and from relationships (9.11) and (9.15).

We note that in the case of "nonresonance" the estimates are improved, specifically

$$\|x(t,\varepsilon) - x^{(m)}(t,\varepsilon)\| \leqslant \varepsilon^m C, \quad \left\|\frac{dx}{dt} - \frac{dx^{(m)}}{dt}\right\| \leqslant \varepsilon^m C. \tag{9.26}$$

In addition to this in the case of "resonance" when the free term is small ($P_0(\tau)\equiv 0$) the estimates may have the form (9.26).

Section 10. The Dynamic Forces in a Visco-Elastic Thread of Variable Length with a Load Q at the End

In Section 5 we considered the problem of finding the forces in a visco-elastic thread (cable) of variable length with a load Q at the end. This problem was reduced to the integration of ordinary linear differential equation (5.1).

It has been shown in the work of Yu. D. Sokolov (ref. 69) that the problem of finding forces in a visco-elastic thread of variable length $l=l(t)$ with a load Q at the end is reduced, under specific conditions, to a system of two ordinary linear differential equations /47

$$l^2(t)\frac{d^2\varphi}{dt^2} = -6[2\gamma+\gamma_1 l(t)]\frac{d\varphi}{dt} - 3(\gamma_1 - v)\frac{d\psi}{dt} -$$
$$-3\beta_1\psi - 6[2\beta+\beta_1 l(t) - v^2]\varphi - 3\left(g - \frac{dv}{dt}\right), \qquad (10.1)$$
$$l(t)\frac{d^2\psi}{dt^2} = 4[3\gamma+\gamma_1 l(t) - vl(t)]\frac{d\varphi}{dt} +$$
$$+2(\gamma_1 - 2v)\frac{d\psi}{dt} + 4[3\beta+\beta_1 l(t) - 2v^2]\varphi + 2\beta_1\psi + 4\left(g - \frac{dv}{dt}\right),$$

where

$$\gamma = \frac{g\alpha}{q}, \quad \gamma_1 = \frac{g\alpha}{Q}, \quad \beta = \frac{gK}{q}, \quad \beta_1 = \frac{gK}{Q},$$
$$K = E\sigma, \quad v = \frac{dl}{dt}, \qquad (10.2)$$

g, q, E, σ, α are the same as in equation (5.1).

In system (10.1) the function $\varphi(t)$ determines the relative elongation of the thread while $\psi(t)$ characterizes the force T_1 which develops at the upper end of the thread because, according to reference 69, we may assume approximately that

$$T_1 \approx K\psi(t). \qquad (10.3)$$

In this section we shall show that system (10.1) may be integrated by the asymptotic method which has been presented in Sections 7 and 8 of this chapter.

We shall assume, as in the solution of equation (5.1), that the load is lifted according to a trapezoidal tachogram (fig. 1), i.e., the law of motion for the load is assigned by equations (5.2), (5.3) and (5.4). Then system (10.1) must be reduced to the form (6.1) for each of the load-lifting regions.

1. Let us consider the first load-lifting region. Introducing the quantities

$$\varphi = \frac{x_1 - x_2}{l_0}, \quad \psi = \frac{4x_2 - 3x_1}{3}, \quad T = \omega_0 t, \quad \tau = \varepsilon T, \tag{10.4}$$

where

$$\omega_0 = \frac{c}{l_0}, \quad c = \sqrt{\frac{Kg}{q}}, \quad \varepsilon = \sqrt[3]{\frac{al_0}{2c^2}}, \tag{10.5}$$

we reduce system (10.1) to the form (6.1) where /48

$$A(\tau, \varepsilon) = \sum_{s=0}^{2} \varepsilon^s A_s(\tau), \quad C(\tau, \varepsilon) = \sum_{s=0}^{2} \varepsilon^s C_s(\tau),$$
$$B(\tau, \varepsilon) = \sum_{s=0}^{4} \varepsilon^s B_s(\tau), \quad P(\tau, \varepsilon) = P_0, \tag{10.6}$$

$$A_0(\tau) = \begin{bmatrix} 1 & -1 \\ -1 & \frac{4}{3} \end{bmatrix}, \quad A_1(\tau) = \begin{bmatrix} -2 & 2 \\ 1 & -\frac{4}{3} \end{bmatrix} \tau^2, \quad A_2 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \tau^4,$$

$$B_0(\tau) = \begin{bmatrix} 3(4+\eta_0) & -2(6+\eta_0) \\ -2(6+\eta_0) & 4\left(3+\frac{1}{3}\eta_0\right) \end{bmatrix},$$

$$B_1(\tau) = \begin{bmatrix} -6 & 6 \\ 4 & -4 \end{bmatrix} \eta_0 \tau^2, \quad B_2(\tau) \equiv B_3(\tau) \equiv 0,$$

$$B_4(\tau) = \begin{bmatrix} -24 & 24 \\ 32 & -32 \end{bmatrix} \tau^2,$$

$$C_0(\tau) = \begin{bmatrix} 3\left(\frac{4\gamma}{l_0} + \gamma_1\right) & -2\left(\frac{6\gamma}{l_0} + \gamma_1\right) \\ -2\left(\frac{6\gamma}{l_0} + \gamma_1\right) & 4\left(\frac{3\gamma}{l_0} + \frac{1}{3}\gamma_1\right) \end{bmatrix} \delta_1^0,$$

$$C_1(\tau) = \begin{bmatrix} -6(\gamma_1\delta_1\tau + 1) & 6\left(\gamma_1\delta_1\tau + \frac{4}{3}\right) \\ 4\gamma_1\delta_1\tau & -4\left(\gamma_1\delta_1\tau + \frac{2}{3}\right) \end{bmatrix} \tau,$$

$$C_2(\tau)=\begin{bmatrix}0 & 0\\ 1 & -1\end{bmatrix}8\tau^2,$$

$$P_0=\frac{p_0}{K}\left(1+\frac{a}{g}\right)\begin{bmatrix}-3\\ 4\end{bmatrix}, \tag{10.7}$$

$$p_0=l_0q,\quad \delta_1^0=\sqrt[3]{\frac{2}{al_0C}},\quad \eta_0=\frac{p_0}{Q}.$$

In order to construct the unknown solution we first find the eigenvalues and the eigenvectors of the matrix $B_0(\tau)$ with respect to the matrix $A_0(\tau)$.

From relationships (6.4) and (6.5) we find

$$\omega_1=6+2\eta_0+2\sqrt{9+3\eta_0+\eta_0^2},$$
$$\omega_2=6+2\eta_0-2\sqrt{9+3\eta_0+\eta_0^2}. \tag{10.8}$$
$$\mu_1=\frac{1}{R_1}\begin{bmatrix}2(6+\eta_0)-\omega_1\\ 3(4+\eta_0)-\omega_1\end{bmatrix},\quad \mu_2=\frac{1}{R_2}\begin{bmatrix}2(6+\eta_0)-\omega_2\\ 3(4+\eta_0)-\omega_2\end{bmatrix},$$

where

$$R_1=\sqrt{\frac{\omega_1^2}{3}-2(4+\eta_0)\omega_1+4(12+6\eta_0+\eta_0^2)},$$
$$R_2=\sqrt{\frac{\omega_2^2}{3}-2(4+\eta_0)\omega_2+4(12+6\eta_0+\eta_0^2)} \tag{10.9}$$

Then, according to formulas of Sections 7 and 8, the asymptotic solution for the system of differential equations (10.1) over the first region describing the lifting of weight Q has the following form in the second approximation

$$l_0\varphi=N_1e^{-(\beta_1t+\beta_2t^2+\beta_3t^3)}\{[\mu_{11}-\mu_{21}+(\mu_{12}-\mu_{22})\times$$
$$\times(\Delta_0t^2+\Delta_1t^4)]\sin(\delta_1t+\delta_2t^3+\delta_3t^5+\theta_1)+$$
$$+(\mu_{22}-\mu_{12})(\Delta_2t+\Delta_3t^2)\cos(\delta_1t+\delta_2t^3+\delta_3t^5+\theta_1)\}+$$
$$+N_2e^{-(\bar\beta_1t+\bar\beta_2t^2+\bar\beta_3t^3)}\{[\mu_{12}-\mu_{22}+(\mu_{11}-\mu_{21})(\bar\Delta_0t^2+\bar\Delta_1t^4)]\times$$
$$\times\sin(\bar\delta_1t+\bar\delta_2t^3+\bar\delta_3t^5+\theta_2)+$$
$$+(\mu_{21}-\mu_{11})(\bar\Delta_2t+\bar\Delta_3t^2)\cos(\bar\delta_1t+\bar\delta_2t^3+\bar\delta_3t^5+\theta_2)\}-\frac{p_0}{2K}\left(1+\frac{a}{g}\right);$$
$$\psi=N_1e^{-(\beta_1t+\beta_2t^2+\beta_3t^3)}\left\{\left[\frac{4}{3}\mu_{21}-\mu_{11}+\right.\right.$$

$$+\left(\frac{4}{3}\mu_{22}-\mu_{12}\right)(\Delta_0 t^2+\Delta_1 t^4)]\sin(\delta_1 t+\delta_2 t^3+\delta_3 t^5+\theta_1)+$$

$$+\left(\mu_{12}-\frac{4}{3}\mu_{22}\right)(\Delta_2 t+\Delta_3 t^3)\cos(\delta_1 t+\delta_2 t^3+\delta_3 t^5+\theta_1)\Big\}+$$

$$+N_2 e^{-(\bar{\beta}_1 t+\bar{\beta}_2 t^2+\bar{\beta}_3 t^3)}\left\{\left[\frac{4}{3}\mu_{22}-\mu_{12}+\left(\frac{4}{3}\mu_{21}-\mu_{11}\right)\times\right.\right. \tag{10.10}$$

$$\times(\bar{\Delta}_0 t^2+\bar{\Delta}_1 t^4)]\sin(\bar{\delta}_1 t+\bar{\delta}_2 t^3+\bar{\delta}_3 t^5+\theta_2)+$$

$$+\left(\mu_{11}-\frac{4}{3}\mu_{21}\right)(\bar{\Delta}_2 t+\bar{\Delta}_3 t^2)\cos(\bar{\delta}_1 t+\bar{\delta}_2 t^3+\bar{\delta}_3 t^5+\theta_2)\Big\}+$$

$$+\frac{Q}{K}\left(1+\frac{a}{g}\right)\left[1+\eta_0\left(1-\frac{at^2}{2l_0}\right)\right],$$

where

$$\beta_1=\frac{g\alpha\left(\frac{1}{3}\omega_1^2-8\omega_1+48+12\eta_0\right)}{2Ql_0R_1^2}.$$

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$$\beta_2=\frac{a\left[24\eta_0^2-12(4+\eta_0)\omega_1+(8-\eta_0)\omega_1^2-\frac{1}{3}\omega_1^3\right]}{8\omega_1 l_0 R_1^2},$$

$$\beta_3=\frac{g\alpha a}{12l_0^2QR_1^4}\left[\frac{1}{9}\omega_1^4-\frac{1}{3}(16+3\eta_0)\omega_1^3+\right.$$

$$+\left(96+40\eta_0+\frac{14}{3}\eta_0^2\right)\omega_1^2-(768+528\eta_0+124\eta_0^2+8\eta_0^3)\omega_1+$$

$$\left.+2304(1+\eta_0)+816\eta_0^2+120\eta_0^3\right],$$

$$\Delta_0=\frac{a\eta_0[\omega_1^2-2\eta_0(\omega_1-\omega_2)-24\eta_0]}{2l_0R_1R_2(\omega_2-\omega_1)}.$$

$$\Delta_1=-\frac{a^2\eta_0}{2l_0^2R_1^3R_2^3(\omega_2-\omega_1)^2}\left\{12\eta_0\left[\frac{\omega_1^3}{3}-(8+\eta_0)\omega_1^2+12(4+\eta_0)\omega_1+\right.\right.$$

$$\left.+24\eta_0^2\right]R_2^2+4[\omega_1^2-2\eta_0(\omega_1-\omega_2)-24\eta_0][\eta_0\omega_2-24\eta_0+$$

$$+3\eta_0^2+3(4+\eta_0)\omega_1]R_1^2+\omega_1(\omega_2-\omega_1)(12+2\eta_0-\omega_2)R_1^2R_2^2\},$$

$$\delta_1=\nu_0-\frac{g^2\alpha^2\left(\frac{1}{3}\omega_1^2-8\omega_1+48+12\eta_0\right)^2}{8Q^2l_0^2\nu_0R_1^4},$$

$$\delta_2=\frac{a\nu_0\left[\frac{1}{3}\omega_1^3-(8+\eta_0)\omega_1+12(4+\eta_0)\omega_1+24\eta_0^2\right]}{12l_0\omega_1R_1^2},$$

$$\delta_3=\frac{a^2\nu_0}{160\omega_1^2l_0^2R_1^4}\left\{4\omega_1\left[\frac{\omega_1^3}{3}-(8+\eta_0)\omega_1^2+12(4+\eta_0)\omega_1+\right.\right.$$

$$\left.+24\eta_0^2\right]\left[\frac{1}{3}\omega_1^2-(8+\eta_0)\omega_1+48-12\eta_0+2\eta_0^2\right]-$$

$$-\left[\frac{\omega_1^3}{3}-(8+\eta_0)\omega_1^2+12(4+\eta_0)\omega_1+24\eta_0^2\right]^2-$$

$$-4\omega_1^2\eta_0R_1^2(\omega_1-12-2\eta_0)+\frac{4\omega_1R_1^2\eta_0(12+3\eta_0-\omega_1)}{R_2^2(\omega_2-\omega_1)}\times$$

$$\times\{16\eta_0+4\eta_0^2-\eta_0(\omega_2+2\omega_1)-4\omega_1\}\{\omega_1^2-2\eta_0(\omega_1-\omega_2)-24\eta_0\}\},$$

$$v_0=\sqrt{\frac{Kga}{l_0\rho_0}},\quad \Delta_2=-\frac{2a\omega_1\eta_0\{\omega_2^2+2(\omega_1-\omega_2)(\eta_0+3)\}}{l_0v_0R_1R_2(\omega_2-\omega_1)^2}, \tag{10.11}$$

$$\Delta_3=\frac{g\alpha\omega_1a}{2l_0^2(\omega_2-\omega_1)^2R_1^3R_2^3Qv_0}\{12\eta_0^2R_2^2\left(\frac{\omega_1^2}{3}-8\omega_1+48+12\eta_0\right)+$$

$$+\eta_0R_1^2\{\omega_1^2-2\eta_0(\omega_1-\omega_2)-24\eta_0\}\left[\frac{\omega_2^2}{3}-8\omega_2+48+12\eta_0\right]-$$

$$-2R_1^2R_2^2(\omega_2-\omega_1)(12+3\eta_0-\omega_2)(\omega_1-12-4\eta_0)\},$$

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N_1, N_2, θ_1 and θ_2 are constants of integration which are determined from the initial conditions while the coefficients $\bar{\beta}_1, \bar{\beta}_2, \bar{\beta}_3, \bar{\Delta}_0, \bar{\Delta}_1, \bar{\delta}_1, \bar{\delta}_2, \bar{\delta}_3, \bar{\Delta}_2, \bar{\Delta}_3$ coincide respectively with the coefficients $\beta_1, \beta_2, \beta_3, \Delta_0, \Delta_1, \delta_1, \delta_2, \delta_3, \Delta_2, \Delta_3$, if in the latter ω_1, R_1 are replaced by ω_2, R_2 and vice versa.

2. We shall now find the solution of the system of differential equations (10.1) in the second region describing the motion of the load Q. For this purpose we introduce new variables by means of the following equation:

$$\varphi=\frac{x_1-x_2}{l_1},\quad \psi=\frac{4x_2-3x_1}{3},$$
$$T=\omega_{01}t,\quad \tau=\varepsilon T. \tag{10.12}$$

where

$$\omega_{01}=\frac{c}{l_1},\quad \varepsilon=\sqrt{\frac{v_0}{c}}. \tag{10.13}$$

System (10.1) is reduced to form (6.1), where

$$A(\tau, \varepsilon) = \sum_{s=0}^{2} \varepsilon^s A_s(\tau), \quad C(\tau, \varepsilon) = \sum_{s=0}^{2} \varepsilon^s C_s(\tau),$$

$$B(\tau, \varepsilon) = \sum_{s=0}^{4} \varepsilon^s B_s(\tau), \quad P(\tau, \varepsilon) = P_0. \qquad (10.14)$$

and

$$A_0(\tau) = \begin{bmatrix} 1 & -1 \\ -1 & \frac{4}{3} \end{bmatrix}, \quad A_1(\tau) = \begin{bmatrix} -2 & 2 \\ 1 & -\frac{4}{3} \end{bmatrix} \tau,$$

$$A_2(\tau) = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \tau^2. \qquad C_0(\tau) =$$

$$= \begin{bmatrix} 3\left(\frac{4\gamma}{l_1} + \gamma_1\right) & -2\left(6\frac{\gamma}{l_1} + \gamma_1\right) \\ -2\left(6\frac{\gamma}{l_1} + \gamma_1\right) & 2\left(6\frac{\gamma}{l_1} + \frac{2}{3}\gamma_1\right) \end{bmatrix} f^{(1)},$$

$$C_1(\tau) = \begin{bmatrix} -3(2\gamma_1 f^{(1)}\tau + 1) & 3\left(2\gamma_1 f^{(1)}\tau + \frac{4}{3}\right) \\ 4\gamma_1 f^{(1)}\tau & -2\left(2\gamma_1 f^{(1)}(\tau) + \frac{2}{3}\right) \end{bmatrix}, \qquad (10.15) \quad /52$$

$$C_2(\tau) = \begin{bmatrix} 0 & 0 \\ 4 & -4 \end{bmatrix} \tau, \quad B_0(\tau) = 3 \begin{bmatrix} 4 + \eta_1 & -\left(4 + \frac{2}{3}\eta_1\right) \\ -\left(4 + \frac{2}{3}\eta_1\right) & 4 + \frac{4}{9}\eta_1 \end{bmatrix},$$

$$B_1(\tau) = \begin{bmatrix} -6 & 6 \\ 4 & -4 \end{bmatrix} \eta_1 \tau, \quad B_2(\tau) \equiv B_3(\tau) \equiv 0,$$

$$B_4(\tau) = \begin{bmatrix} -6 & 6 \\ 8 & -8 \end{bmatrix}, \quad P_0 = \frac{P_1}{K} \begin{bmatrix} -3 \\ 4 \end{bmatrix},$$

$$f^{(1)} = \frac{1}{\sqrt{v_0 C}}, \quad p_1 = l_1 q, \quad \eta_1 = \frac{p_*}{Q}.$$

In this case the solution has the form

$$l_1\varphi = N_1^{(1)} e^{-(\beta_1^{(1)}t + \beta_2^{(1)}t^2)} \{[\mu_{11}^{(1)} - \mu_{21}^{(1)} + (\mu_{12}^{(1)} - \mu_{22}^{(1)})(\Delta_0^{(1)} + \Delta_1^{(1)}t^2)] \times$$
$$\times \sin(\delta_1^{(1)}t + \delta_2^{(1)}t^2 + \delta_3^{(1)}t^3 + \theta_1^{(1)}) + (\mu_{22}^{(1)} - \mu_{12}^{(1)})(\Delta_2^{(1)} + \Delta_3^{(1)}t) \times$$
$$\times \cos(\delta_1^{(1)}t + \delta_2^{(1)}t^2 + \delta_3^{(1)}t^3 + \theta_1^{(1)})\} +$$
$$+ N_2^{(1)} e^{-(\bar{\beta}_1 t + \bar{\beta}_2 t^2)} \{[\mu_{12}^{(1)} - \mu_{22}^{(1)} + (\mu_{11}^{(1)} - \mu_{21}^{(1)})(\bar{\Delta}_0^{(1)}t + \bar{\Delta}_1^{(1)}t^2)] \times$$
$$\times \sin(\bar{\delta}_1^{(1)}t + \bar{\delta}_2^{(1)}t^2 + \bar{\delta}_3^{(1)}t^3 + \theta_2^{(1)}) +$$

$$+ (\mu_{21}^{(1)} - \mu_{11}^{(1)})(\bar{\Delta}_2^{(1)} + \bar{\Delta}_3^{(1)} t)\cos(\bar{\delta}_1^{(1)} t + \bar{\delta}_2^{(1)} t^2 + \bar{\delta}_3^{(1)} t^3 + \theta_2^{(1)})\} - \frac{p_1}{2K}$$

$$\psi = N_1^{(1)} e^{-(\beta_1^{(1)} t + \beta_2^{(1)} t^2)} \left\{ \left[\frac{4}{3}\mu_{21}^{(1)} - \mu_{11}^{(1)} + \right.\right.$$

$$+ \left(\frac{4}{3}\mu_{22}^{(1)} - \mu_{12}^{(1)} \right)(\Delta_0^{(1)} t + \Delta_1^{(1)} t^2)] \sin(\delta_1^{(1)} t + \delta_2^{(1)} t^2 + \delta_3^{(1)} t^3 + \theta_1^{(1)}) +$$

$$\left. + \left(\mu_{12}^{(1)} - \frac{4}{3}\mu_{22}^{(1)} \right)(\Delta_2^{(1)} + \Delta_3^{(1)} t)\cos(\delta_1^{(1)} t + \delta_2^{(1)} t^2 + \delta_3^{(1)} t^3 + \theta_1^{(1)}) \right\} + \qquad (10.16)$$

$$+ N_2^{(1)} e^{-(\bar{\beta}^{(1)} t + \bar{\beta}_2^{(1)} t^2)} \left\{ \left[\frac{4}{3}\mu_{22}^{(1)} - \mu_{12}^{(1)} + \right.\right.$$

$$+ \left(\frac{4}{3}\mu_{21}^{(1)} - \mu_{11}^{(1)} \right)(\bar{\Delta}_0^{(1)} t + \bar{\Delta}_1^{(1)} t^2) \Big] \sin(\bar{\delta}_1^{(1)} t + \bar{\delta}_2^{(1)} t^3 + \theta_2^{(1)}) +$$

$$\left. + \left(\mu_{11}^{(1)} - \frac{4}{3}\mu_{21}^{(1)} \right)(\bar{\Delta}_2^{(1)} + \bar{\Delta}_3^{(1)} t)\cos(\bar{\delta}_1^{(1)} t + \bar{\delta}_2^{(1)} t^2 + \bar{\delta}_3^{(1)} t^3 + \theta_2^{(1)}) \right\} +$$

$$+ \frac{Q}{K}\left[1 + \eta_1 \left(1 - \frac{v_0 t}{l_1} \right) \right],$$

where

$$\beta_1^{(1)} = \frac{g\alpha \left[\frac{1}{3}(\omega_1^{(1)})^2 - 8\omega_1^{(1)} + 48 + 12\eta_1 \right]}{2Ql_1 (R_1^{(1)})^2} + \frac{v_0}{4l_1 \omega_1^{(1)} (R_1^{(1)})^2} \times$$

$$\times \left[24\eta_1^2 + 12\eta_1 \omega_1^{(1)} - 48\omega_1^{(1)} - \eta_1 (\omega_1^{(1)})^2 + 8(\omega_1^{(1)})^2 - \frac{1}{3}(\omega_1^{(1)})^3 \right],$$

$$\beta_2^{(1)} = \frac{v_0 g\alpha}{4l_1^2 Q (R_1^{(1)})^4} \left\{ \left[\frac{1}{3}(\omega_1^{(1)})^2 - 8\omega_1^{(1)} + 48 + 12\eta_1^2 \right] \left[\frac{1}{3}(\omega_1^{(1)})^2 - \right.\right.$$

$$\left. - \eta_1 \omega_1^{(1)} - 8\omega_1^{(1)} + 48 + 12\eta_1 + 2\eta_1^2 \right] - 2\eta_1 (R_1^{(1)})^2 (\omega_1^{(1)} - 12) \Big\},$$

$$\Delta_0^{(1)} = \frac{v_0 \eta_1 \left| (\omega_1^{(1)})^2 - 2\eta_1 (\omega_1^{(1)} - \omega_2^{(1)}) - 24\eta_1 \right|}{l_1 R_1^{(1)} R_2^{(1)} (\omega_2^{(1)} - \omega_1^{(1)})},$$

$$\Delta_1^{(1)} = -\frac{v_0^2 \eta_1}{l_1^2 (R_1^{(1)})^3 (R_2^{(1)})^3 (\omega_2^{(1)} - \omega_1^{(1)})^2} \left\{ 12\eta_1 \left[\frac{1}{3}(\omega_1^{(1)})^3 - (8 + \eta_1)(\omega_1^{(1)})^2 + \right.\right.$$

$$+ 12(4 + \eta_1)\omega_1^{(1)} + 24\eta_1^2 \Big] (R_2^{(1)})^2 + \left| (\omega_1^{(1)})^2 - 2\eta_1 (\omega_1^{(1)} - \omega_2^{(1)}) - \right.$$

$$- 24\eta_1 \Big| \left| 4\eta_1 \omega_2^{(1)} - 96\eta_1 + 12\omega_1^{(1)} (4 + \eta_1) \right| (R_1^{(1)})^2 +$$

$$\left. + \omega_1^{(1)} (\omega_2^{(1)} - \omega_1^{(1)}) (12 + 2\eta_1 - \omega_2^{(1)}) \left| R_2^{(1)} R_1^{(1)} \right|^2 \right\},$$

$$\Delta_2^{(1)} = -\frac{2v_0 \omega_1^{(1)} \eta_1 \left| (\omega_2^{(1)})^2 + 2(\omega_2^{(1)} - \omega_1^{(1)})(3 + \eta_1) \right|}{l_1 v^{(1)} (\omega_2^{(1)} - \omega_1^{(1)})^2 R_1^{(1)} R_2^{(1)}},$$

$$\Delta_3^{(1)} = \frac{v_0 g\alpha\omega_1^{(1)}}{l_1^2 Q\nu^{(1)}|R_1^{(1)}R_2^{(1)}|^3(\omega_2^{(1)}-\omega_1^{(1)})^2}\left\{12\eta_1^2(R_1^{(1)})^2\left|\frac{1}{3}(\omega_1^{(1)})^2-\right.\right.$$

$$-8\omega_1^{(1)}+48+12\eta_1\Big| + \eta_1(R_1^{(1)})^2\left[\frac{1}{3}(\omega_2^{(1)})^2-8\omega_2^{(1)}+48+12\eta_1\right]\times$$

$$\times\left|(\omega_1^{(1)})^2-2\eta_1(\omega_1^{(1)}-\omega_2^{(1)})-24\eta_1\right|-2(\omega_2^{(1)}-\omega_1^{(1)})(R_1^{(1)})^2\times$$

$$\times(12+3\eta_1-\omega_2^{(1)})(\omega_2^{(1)}-12-4\eta_1)\Big\},$$

$$\delta_1^{(1)} = \nu^{(1)} - \frac{g^2\alpha^2\left[\frac{1}{3}(\omega_1^{(1)})^2-8\omega_1^{(1)}+48+12\eta_1\right]^2}{8\nu^{(1)}l_1^2Q^2(R_1^{(1)})^4},$$

$$\delta_2^{(1)} = -\frac{v_0\nu^{(1)}\left[\frac{1}{3}(\omega_1^{(1)})^3-(8+\eta_1)(\omega_1^{(1)})^2+(48+12\eta_1)\omega_1^{(1)}+24\eta_1^2\right]}{4l_1^2\omega_1^{(1)}(R_1^{(1)})^2},$$

$$\delta_3^{(1)} = -\frac{v_0\nu^{(1)}}{24l_1^2(\omega_1^{(1)})^2(R_1^{(1)})^4}\left\{\frac{1}{3}(\omega_1^{(1)})^3-(8+\eta_1)(\omega_1^{(1)})^2+\right.$$

$$+12(4+\eta_1)\omega_1^{(1)}+24\eta_1^2-4\omega_1^{(1)}\left[\frac{1}{3}(\omega_1^{(1)})^3-(8+\eta_1)(\omega_1^{(1)})^2+\right.$$

$$\left.+12(4+\eta_1)\omega_1^{(1)}+24\eta_1^2\right]\left[\frac{1}{3}(\omega_1^{(1)})^2-(8+\eta_1)\omega_1^{(1)}+\right.$$

$$\left.+12(4+\eta_1)+2\eta_1^2\right|+4(\omega_1^{(1)})^2\eta_1(R_1^{(1)})^2(\omega_1^{(1)}-12-2\eta_1)-$$

$$-\frac{4\omega_1^{(1)}\eta_1(R_1^{(1)})^2(12+3\eta_1-\omega_1^{(1)})}{(R_2^{(1)})^2(\omega_2^{(1)}-\omega_1^{(1)})}\left|16\eta_1+4\eta_1^2-\eta_1(\omega_2^{(1)}+2\omega_1^{(1)})-\right.$$

$$\left.-4\omega_1^{(1)}\right]\left[(\omega_1^{(1)})^2-2\eta_1(\omega_1^{(1)}-\omega_2^{(1)})-24\eta_1\right]\Big\}. \qquad (10.17)$$

$N_1^{(1)}, N_2^{(1)}, \theta_1^{(1)}, \theta_2^{(1)}$, are constants of integration which are determined from the initial conditions, while the coefficients $\nu^{(1)}, \bar{\beta}_1^{(1)}, \bar{\beta}_2^{(1)}, \bar{\Delta}_0^{(1)}, \bar{\Delta}_1^{(1)}, \bar{\delta}_1^{(1)}, \bar{\delta}_2^{(1)}$, $\bar{\delta}_3^{(1)}, \bar{\Delta}_2^{(1)}, \bar{\Delta}_3^{(1)}$ coincide respectively with coefficients $\nu, \beta_1^{(1)}, \beta_2^{(1)}, \Delta_0^{(1)}, \Delta_1^{(1)}$, $\delta_1^{(1)}, \delta_2^{(1)}, \delta_3^{(1)}, \Delta_2^{(1)}, \Delta_3^{(1)}$, if in the latter we replace $t_0, \omega_1^{(1)}, R_1^{(1)}, p_0$ by t_1, /55 $\omega_2^{(1)}, R_2^{(1)}$; $\omega_2^{(1)}, R_2^{(1)}$ and $\omega_1^{(1)}, R_1^{(1)}$, where $\omega_1^{(1)}, \omega_2^{(1)}, \mu_{11}^{(1)}, \mu_{21}^{(1)}, \mu_{12}^{(1)}, \mu_{22}^{(1)}, R_1^{(1)}$, $R_2^{(1)}$ are determined from (10.8) and (10.9), in which η_0 must be replaced by

$$\eta_1 = \frac{p_1}{Q}, \quad p_1 = l_1 q. \qquad (10.18)$$

3. In order to integrate the system of differential equations (10.1) in the third region, defining the lifting of load Q, we must, as before, represent this system in form (6.1). For this purpose we introduce the new variables

$$\varphi = \frac{x_1 - x_2}{l_2}, \quad \psi = \frac{4x_2 - 3x_1}{3}, \quad T = \omega_{02} t, \quad \tau = \varepsilon T. \tag{10.19}$$

where

$$\omega_{02} = \frac{c}{l_2}, \quad \varepsilon = \sqrt{\frac{V_0}{c}}. \tag{10.20}$$

Then

$$A(\tau, \varepsilon) = \sum_{s=0}^{\cdot} \varepsilon^s A_s(\tau), \quad C(\tau, \varepsilon) = \sum_{s=0}^{\cdot} \varepsilon^s C_s(\tau), \tag{10.21}$$

$$B(\tau, \varepsilon) = \sum_{s=0}^{6} \varepsilon^s B_s(\tau), \quad P(\tau, \varepsilon) = P_0,$$

where

$$A_0(\tau) = \begin{bmatrix} 1 & -1 \\ -1 & \frac{4}{3} \end{bmatrix}, \quad A_1(\tau) = \begin{bmatrix} -2 & 2 \\ 1 & -\frac{4}{3} \end{bmatrix} \tau,$$

$$A_2(\tau) = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \tau^2, \quad A_3(\tau) = 2 \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \delta_3^* \tau^3,$$

$$A_4(\tau) = (\delta_3^*)^2 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \tau^4, \tag{10.22}$$

$$C_0(\tau) = f^{(1)} \begin{bmatrix} 3\left(\frac{4\gamma}{l_2} + \gamma_1\right) & -3\left(\frac{4\gamma}{l_2} + \frac{2}{3}\gamma_1\right) \\ -3\left(\frac{4\gamma}{l_2} + \frac{2}{3}\gamma_1\right) & 2\left(\frac{6\gamma}{l_2} + \frac{2}{3}\gamma_1\right) \end{bmatrix}$$

$$C_1(\tau) = \begin{bmatrix} -3(1 + 2f^{(1)}\gamma_1\tau) & 3\left(2\gamma_1 f^{(1)}\tau + \frac{4}{3}\right) \\ 4\gamma_1 f^{(1)}\tau & -2\left(2\gamma_1 f^{(1)}\tau + \frac{2}{3}\right) \end{bmatrix}, \tag{56}$$

$$C_2(\tau) = \begin{bmatrix} 6(\delta_3^*\gamma_1 f^{(1)}\tau^2 + \delta_3^*\tau) & -6\left(\delta_3^* f^{(1)}\tau^2 + \frac{4}{3}\delta_3^*\tau\right) \\ -2(2\gamma_1 f^{(1)}\delta_3^*\tau^2 - 2\tau) & 4\left(\gamma_1 f^{(1)}\delta_3^*\tau^2 - \tau + \frac{2}{3}\delta_3^*\tau\right) \end{bmatrix},$$

$$C_3(\tau) = 6\delta_3^* \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \tau^2, \quad C_4(\tau) = 4(\delta_3^*)^2 \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} \tau^2,$$

$$B_0(\tau)=\begin{bmatrix} 3(4+\eta_2) & -3\left(4+\frac{2}{3}\eta_2\right) \\ -3\left(4+\frac{2}{3}\eta_2\right) & 4\left(3+\frac{1}{3}\eta_2\right)\end{bmatrix},\quad B_1(\tau)=2\eta_2\tau\begin{bmatrix}-3 & 3\\ 2 & -2\end{bmatrix},$$

$$B_2(\tau)=2\eta_2\delta_3^*\tau^2\begin{bmatrix}3 & -3\\ -2 & 2\end{bmatrix},\quad B_4(\tau)=2\begin{bmatrix}-3 & 3\\ 4 & -4\end{bmatrix},\quad B_3(\tau)\equiv 0,$$

$$B_5(\tau)=8\delta_3^*\tau\begin{bmatrix}-3 & 3\\ 4 & -4\end{bmatrix},\quad B_6(\tau)=8(\delta_3^*)^2\tau^2\begin{bmatrix}-3 & 3\\ 4 & -4\end{bmatrix},$$

$$P_0=\frac{p_2}{K}\left(1-\frac{a}{g}\right)\begin{bmatrix}-3\\ 4\end{bmatrix},\quad \eta_2=\frac{p_2}{Q},\quad p_2=l_2q,\quad \delta_3^*=\frac{al_2}{2v_0^2}.$$

In the third region the solution of system (10.1), in the second approximation, will have the form

$$\begin{aligned}
l_2\varphi &= N_1^{(2)}e^{-(\beta_1^{(2)}t+\beta_2^{(2)}t^2)}\{[\mu_{11}^{(2)}-\mu_{21}^{(2)}+\\
&+(\mu_{12}^{(2)}-\mu_{22}^{(2)})(\Delta_0^{(2)}t+\Delta_1^{(2)}t^2)\sin(\delta_1^{(2)}t+\delta_2^{(2)}t^2+\delta_3^{(2)}t^3+\theta_1^{(2)})+\\
&+(\mu_{22}^{(2)}-\mu_{12}^{(2)})(\Delta_2^{(2)}+\Delta_3^{(2)}t)\cos(\delta_1^{(2)}t+\delta_2^{(2)}t^2+\delta_3^{(2)}t^3+\theta_1^{(2)})\}+\\
&+N_2^{(2)}e^{-(\bar\beta_1^{(2)}t+\bar\beta_2^{(2)}t^2)}\{[\mu_{12}^{(2)}-\mu_{22}^{(2)}+(\mu_{11}^{(2)}-\mu_{21}^{(2)})(\bar\Delta_0^{(2)}t+\bar\Delta_1^{(2)}t^2)]\times\\
&\times\sin(\bar\delta_1^{(2)}t+\bar\delta_2^{(2)}t^2+\bar\delta_3^{(2)}t^3+\theta_2^{(2)})+\\
&+(\mu_{21}^{(2)}-\mu_{11}^{(2)})(\bar\Delta_2^{(2)}+\bar\Delta_3^{(2)}t)\cos(\bar\delta^{(2)}t+\bar\delta_2^{(2)}t^2+\bar\delta_3^{(2)}t^3+\theta_2^{(2)})\}-\frac{p_2}{2K}\left(1-\frac{a}{g}\right),
\end{aligned}\tag{10.23}$$

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$$\begin{aligned}
\psi &= N_1^{(2)}e^{-(\beta_1^{(2)}t+\beta_2^{(2)}t^2)}\left\{\frac{4}{3}\mu_{21}^{(2)}-\mu_{11}^{(2)}+\left(\frac{4}{3}\mu_{22}^{(2)}-\mu_{12}^{(2)}\right)(\Delta_0^{(2)}t+\Delta_1^{(2)}t^2)\times\right.\\
&\times\sin(\delta_1^{(2)}t+\delta_2^{(2)}t^2+\delta_3^{(2)}t^3+\theta_1^{(2)})+\left(\mu_{12}^{(2)}-\frac{4}{3}\mu_{22}^{(2)}\right)(\Delta_2^{(2)}+\Delta_3^{(2)}t)\times\\
&\left.\times\cos(\delta_1^{(2)}t+\delta_2^{(2)}t^2+\delta_3^{(2)}t^3+\theta_1^{(2)})\right\}+N_2^{(2)}e^{-(\bar\beta_1^{(2)}t+\bar\beta_2^{(2)}t^2)}\times\\
&\times\left\{\left|\frac{4}{3}\mu_{22}^{(2)}-\mu_{12}^{(2)}+\left(\frac{4}{3}\mu_{21}^{(2)}-\mu_{11}^{(2)}\right)(\bar\Delta_0^{(2)}t+\bar\Delta_1^{(2)}t^2)\right|\sin(\bar\delta_1^{(2)}t+\right.\\
&+\bar\delta_2^{(2)}t^2+\bar\delta_3^{(2)}t^3+\theta_2^{(2)})+\left(\mu_{11}^{(2)}-\frac{4}{3}\mu_{21}^{(2)}\right)(\Delta_2^{(2)}+\bar\Delta_3^{(2)}t)\times\\
&\left.\times\cos(\bar\delta_1^{(2)}t+\bar\delta_2^{(2)}t^2+\bar\delta_3^{(2)}t^3+\theta_2^{(2)})\right\}+\frac{Q}{K}\left(1-\frac{a}{g}\right)\times\\
&\times\left[1+\eta_2\left(1-\frac{v_0}{l_2}t+\frac{a}{2l_2}t^2\right)\right],
\end{aligned}$$

where $N_1^{(2)}, N_2^{(2)}, \theta_1^{(2)}, \theta_2^{(2)}$ are constants of integration while the coefficients

$\beta_1^{(2)}$, $\beta_2^{(2)}$, $\Delta_0^{(2)}$, $\delta_1^{(2)}$, $\delta_2^{(2)}$, $\Delta_2^{(2)}$, $\Delta_3^{(2)}$, $\bar\beta_1^{(2)}$, $\bar\beta_2^{(2)}$, $\bar\Delta_0^{(2)}$, $\bar\delta_1^{(2)}$, $\bar\delta_2^{(2)}$, $\bar\Delta_2^{(2)}$, $\bar\Delta_3^{(2)}$, coincide respectively

with the coefficients $\beta_1^{(1)}$, $\beta_2^{(1)}$, $\Delta_0^{(1)}$, $\delta_1^{(1)}$, $\delta_2^{(1)}$, $\Delta_2^{(1)}$, $\Delta_3^{(1)}$, $\bar{\beta}_1^{(1)}$, $\bar{\beta}_2^{(1)}$, $\bar{\Delta}_0^{(1)}$, $\bar{\delta}_1^{(1)}$, $\bar{\delta}_2^{(1)}$, $\bar{\Delta}_2^{(1)}$, $\bar{\Delta}_3^{(1)}$, if in the latter the quantities η_1, $\omega_1^{(1)}$, $\omega_2^{(1)}$, $R_1^{(1)}$, $R_2^{(1)}$ are replaced respectively with the quantities η_2, $\omega_1^{(2)}$, $\omega_2^{(2)}$, $R_1^{(2)}$, $R_2^{(2)}$, which are determined from relationships (10.8), (10.9) with a preliminary replacement of quantity η_0 by η_2.

The coefficients $\Delta_1^{(2)}$, $\delta_3^{(2)}$ have the form

$$\Delta_1^{(2)} = -\frac{v_0\eta_2}{l_2^2(R_1^{(2)}R_2^{(2)})^3(\omega_2^{(2)}-\omega_1^{(2)})^2}\left\{12\eta_2^2\left[\frac{1}{3}(\omega_1^{(2)})^3 - (8+\eta_2)(\omega_1^{(2)})^2 + 12(4+\eta_2)\omega_1^{(2)} + 24\eta_2^{(2)}\right](R_2^{(2)})^2 + \right.$$
$$+ (R_1^{(2)})^2[(\omega_1^{(2)})^2 - 2\eta_2(\omega_1^{(2)}-\omega_2^{(2)}) - 24\eta_2][4\eta_2\omega_2^{(2)} - 96\eta_2 +$$
$$+ 12\omega_1^{(2)}(4+\eta_2)] + \omega_1^{(2)}(R_1^{(2)}R_2^{(2)})^2(\omega_2^{(2)}-\omega_1^{(2)})(12+2\eta_2-\omega_2^{(2)}) -$$
$$- \omega_1^{(2)}(R_1^{(2)}R_2^{(2)})^2\delta_3^*(\omega_2^{(2)}-\omega_1^{(2)})(2\omega_2^{(2)}\eta_2 + \omega_1^{(2)}\eta_2 - 24\eta_2 - 6\eta_2^2) +$$
$$\left. + 2\delta_3^*\eta_2^2(R_1^{(2)}R_2^{(2)})^2(\omega_2^{(2)}-\omega_1^{(2)})[4\omega_2^{(2)} + 3\omega_1^{(2)} - 36 - 12\eta_2]\right\}$$

$$\delta_3^{(2)} = -\frac{v_0^2\nu^2}{24l_2^2(\omega_1^{(2)})^2(R_1^{(2)})^4}\left\{\left[\frac{1}{3}(\omega_1^{(2)})^3 - (8+\eta_2)(\omega_1^{(2)})^2 + \right.\right.$$
$$\left. + 12(4+\eta_2)\omega_1^{(2)} + 24\eta_2^2\right]^2 - 4\omega_1^{(2)}\left[\frac{1}{3}(\omega_1^{(2)})^3 - (8+\eta_2)(\omega_1^{(2)})^2 + \right. \qquad (10.24)$$
$$\left. + 12(4+\eta_2)\omega_1^{(2)} + 24\eta_2^2\right]\left[\frac{1}{3}(\omega_1^{(2)})^2 - (8+\eta_2)\omega_1^{(2)} + 12(4+\eta_2) + \right.$$
$$\left. + 2\eta_2^2\right] + 4\eta_2(\omega_1^{(2)}R_1^{(2)})^2(\omega_1^{(2)} - 12 - 2\eta_2) - \qquad /58$$
$$- \frac{4\eta_2\omega_1^{(2)}(R_1^{(2)})^2(12+3\eta_2-\omega_1^{(2)})}{(R_2^{(2)})^2(\omega_2^{(2)}-\omega_1^{(2)})}[4\eta_2^2 - \eta_2(\omega_2^{(2)} + 2\omega_1^{(2)} - 16) - 4\omega_1^{(2)}] \times$$
$$\times [(\omega_1^{(2)})^2 - 2\eta_2(\omega_1^{(2)} - \omega_2^{(2)} + 12)] +$$
$$+ 4\omega_1^{(2)}(R_1^{(2)})^2\delta_3^*\left[\frac{1}{3}(\omega_1^{(2)})^2 - (8+\eta_2)\omega_1^{(2)} + 12(4+\eta_2) + 2\eta_2^2\right] -$$
$$\left. - 8\omega_1^{(2)}\delta_3^*(\eta_2R_1^{(2)})^2(\omega_1^{(2)} - 12)\right\},$$

$\nu^{(2)}$ coincides with $\nu^{(1)}$ if in the latter we replace the quantities $\omega_1^{(1)}$, ι_1, p_1 by the quantities $\omega_1^{(2)}$, ι_2, p_2.

In order to obtain the constant coefficients $\bar{\delta}_3^{(2)}, \bar{\Delta}_1^{(2)}$, it is necessary to use $\omega_2^{(2)}, R_2^{(2)}$ in place of $\omega_1^{(2)}, R_1^{(2)}$ and $\omega_1^{(2)}, R_1^{(2)}$ in place of $\omega_2^{(2)}, R_2^{(2)}$ in coefficients $\delta_3^{(2)}, \Delta_1^{(2)}$.

The eigenfunctions $\mu_{11}^{(2)}, \mu_{21}^{(2)}, \mu_{12}^{(2)}, \mu_{22}^{(2)}$ are determined by relationships (10.8) where $\eta_0, \omega_1, \omega_2, R_1, R_2$ are replaced respectively by the quantities $\eta_2, \omega_1^{(2)}, \omega_2^{(2)}, R_1^{(2)}, R_2^{(2)}$.

4. Proceeding from the asymptotic solution (10.16), for a given coefficient α, we can find a limiting value of the velocity v_{lim} ($v<v_{lim}$), for which the forces T_1 will be damped out. This relationship has the form

$$q > \frac{vQ}{2g}\left\{1 + \frac{\eta_1 |(\omega_2^{(1)})^2 - 24\omega_2^{(1)} - 24\eta_1|}{\frac{1}{3}(\omega_2^{(1)})^3 + 48\omega_2^{(1)} + 12\eta_1\omega_2^{(1)} - 8(\omega_2^{(1)})^2}\right\}. \qquad (10.25)$$

To illustrate the nature of variation in the dynamic forces of a viscoelastic thread during the first stage, we have used the above equations to compute the forces when the various parameters are as follows:

$$\begin{aligned} &Q=20.10 \text{ kg-wt}, \quad \eta_0=0.555, \quad \alpha=3.10 \text{ kg-wt}\cdot\text{sec}, \\ &a=0.1 \text{ g}, \quad v_0=14.028, \quad K=20791284 \text{ kg-wt}, \\ &q=11.5625 \frac{\text{kg-wt}}{\text{meter}}, \quad l_0=1200 \text{ meter}. \end{aligned} \qquad (10.26)$$

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The nature of variation in the forces which occur at the upper end of the thread during the first stage motion of load Q is shown in Table 1. From this table we can see that

$$\max T_1 = 56502 \text{ kg-wt}.$$

If we use the equation proposed by G. N. Savin (ref. 62), max T_1 is approximately equal to 56,300 kilograms (weight) when the numerical values of the parameters are the same.

Consequently the discrepancy in the numerical results which have been obtained does not exceed 0.36 percent.

Section 11. Boundary Value Problem for a System of Linear Differential Equations of the Second Order

In the present section we shall apply the asymptotic method presented above (Sections 7 and 8) to find the large eigenvalues of the boundary value problem which is determined by a system of differential equations

$$A(t,\lambda)\frac{d^2x}{dt_1^2}+C(t_1,\lambda)\frac{dx}{dt_1}+\lambda B(t_1,\lambda)x=0 \tag{11.1}$$

and by some regular boundary conditions

$$\begin{aligned} G_1x'(1)+F_1x'(0)+G_0x(1)+F_0x(0)=0,\\ K_1x'(1)+M_1x'(0)+K_0x(1)+M_0x(0)=0, \end{aligned} \tag{11.2}$$

where $x(t_1)$ is an n-dimensional vector and $G_i, F_i, K_i, M_i (i=0, 1)$ are constant real matrices of the n-th order.

We let

$$A(t_1,\lambda)=\sum_{s=0}^{\infty}\lambda^{-\frac{s}{2}}A_s(t_1),\quad B(t_1,\lambda)=\sum_{s=0}^{\infty}\lambda^{-\frac{s}{2}}B_s(t_1),$$
$$C(t_1,\lambda)=\sum_{s=0}^{\infty}\lambda^{-\frac{s}{2}}C_s(t_1).$$

Assuming that

$$t_1=\varepsilon t=\tau,\quad \varepsilon=\frac{1}{\sqrt{\lambda}},$$

we transform system (11.1) to the form (7.27):

$$A(\tau,\varepsilon)\frac{d^2x}{dt^2}+\varepsilon C(\tau,\varepsilon)\frac{dx}{dt}+B(\tau,\varepsilon)x=0. \tag{11.3}$$ /60

Here $A(\tau,\varepsilon), B(\tau,\varepsilon), C(\tau,\varepsilon)$ are matrices which satisfy the conditions of Section 6. In view of the calculation steps carried in Section 7, the particular solutions for the system of equations (11.3) have the form /61

$$\begin{aligned} x_\nu(\tau,\varepsilon)=\Pi_\nu(\tau,\varepsilon)e^{\frac{1}{\varepsilon}\int_0^\tau D_\nu(\tau,\varepsilon)d\tau}\Bigg[C_{1\nu}\cos\frac{1}{\varepsilon}\int_0^\tau\Omega_\nu(\tau,\varepsilon)d\tau+\\ +C_{2\nu}\sin\frac{1}{\varepsilon}\int_0^\tau\Omega_\nu(\tau,\varepsilon)d\tau\Bigg], \end{aligned} \tag{11.4}$$

TABLE 1.

Time	Forces	Time	Forces	Time	Forces
0	39352,620	4,8	30479,357	9,6	29210,112
0,1	42573,397	4,9	32284,290	9,7	28707,524
0,2	45788,894	5,0	34655,016	9,8	29087,609
0,3	49021,606	5,1	37592,388	9,9	30332,945
0,4	52001,039	5,2	40923,297	10,0	32281,005
0,5	54301,511	5,3	44309,865	10,1	34771,094
0,6	55731,732	5,4	47476,980	10,2	37717,260
0,7	56422,576	5,5	50332,247	10,3	40988,291
0,8	56501,583	5,6	52825,185	10,4	44377,104
0,9	55845,971	5,7	54769,648	10,5	47553,284
1,0	54266,623	5,8	55913,501	10,6	50343,786
1,1	51825,152	5,9	56154,410	10,7	52657,586
1,2	48965,116	6,0	55594,147	10,8	54366,837
1,3	45899,545	6,1	54355,548	10,9	55288,120
1,4	42687,063	6,2	52443,353	11,0	55342,510
1,5	39318,480	6,3	49849,307	11,1	54513,562
1,6	37279,770	6,4	46737,206	11,2	52985,818
1,7	33239,566	6,5	43417,648	11,3	50792,171
1,8	31179,940	6,6	40142,418	11,4	47994,059
1,9	29804,763	6,7	37012,561	11,5	44777,627
2,0	29030,267	6,8	34119,724	11,6	41421,415
2,1	28980,409	6,9	31683,526	11,7	38147,349
2,2	29860,796	7,0	29969,118	11,8	35088,577
2,3	31648,680	7,1	29090,894	11,9	32402,821
2,4	34056,040	7,2	28990,618	12,0	30320,574
2,5	36813,442	7,3	29611,300	12,1	29023,967
2,6	39868,327	7,4	30993,338	12,2	28549,198
2,7	43221,337	7,5	33151,370	12,3	28862,502
2,8	46660,444	7,6	35923,347	12,4	29972,424
2,9	49797,745	7,7	39030,001	12,5	31878,007
3,0	52361,788	7,8	42260,509	12,6	34447,935
3,1	54336,129	7,9	45517,610	12,7	37447,681
3,2	55753,387	8,0	48668,154	12,8	40678,168
3,3	56490,459	8,1	51453,105	12,9	44001,031
3,4	56313,671	8,2	53607,145	13,0	47226,009
3,5	55195,037	8,3	55020,619	13,1	50084,082
3,6	53352,077	8,4	55710,183	13,2	52354,948
3,7	51015,012	8,5	55656,230	13,3	53945,086
3,8	48238,108	8,6	54770,001	13,4	54796,801
3,9	45029,316	8,7	53041,559	13,5	54809,567
4,0	41562,733	8,8	50637,484	13,6	53926,561
4,1	38216,958	8,9	47792,571	13,7	52236,770
4,2	35269,503	9,0	44657,890	13,8	49905,672
4,3	32784,362	9,1	41327,916	13,9	47057,204
4,4	30770,061	9,2	37987,901	14,0	43812,225
4,5	29383,199	9,3	34929,004	14,1	40394,533
4,6	28861,296	9,4	32393,694	14,2	37086,889
4,7	29278,078	9,5	30474,617	14,3	34095,294

Note: Commas in Table 1 represent decimal points.

where $C_{1\nu}, C_{2\nu} (\nu=1, 2, \ldots, n)$ are constants of integration.

The general solution of system (11.1) will be written in the following manner

$$X(\tau,\varepsilon) = X_1(\tau, \varepsilon)C_1 + X_2(\tau, \varepsilon)C_2, \qquad (11.5)$$

where $X_1(\tau,\varepsilon), X_2(\tau,\varepsilon)$ are the fundamental matrices of the solutions of system (11.1) having the form

$$X_1(\tau,\varepsilon)=\left[\Pi_\nu(\tau,\varepsilon)e^{\frac{1}{\varepsilon}\int_0^\tau D_\nu(\tau,\varepsilon)d\tau}\cos\frac{1}{\varepsilon}\int_0^\tau \Omega_\nu(\tau,\varepsilon)\,d\tau\right],$$

$$X_2(\tau,\varepsilon)=\left[\Pi_\nu(\tau,\varepsilon)e^{\frac{1}{\varepsilon}\int_0^\tau D_\nu(\tau,\varepsilon)d\tau}\sin\frac{1}{\varepsilon}\int_0^\tau \Omega_\nu(\tau,\varepsilon)\,d\tau\right] \quad (\nu=1,2,\ldots,n), \tag{11.6}$$

C_1, C_2 are constant n-dimensional vectors which are determined from the boundary conditions (11.2).

By substituting the solution $X(\tau,\varepsilon)$ into the boundary conditions (11.2) we obtain a system of algebraic equations with respect to C_1 and C_2. As we know, in order that a non-trivial solution of this system exist it is necessary and sufficient that its determinant be equal to zero. By setting the determinant equal to zero we find a transcendental equation for determining the eigenvalues of the boundary value problem.

As an example we shall consider a case which is frequently encountered in practice, when the boundary conditions (11.2) assume the form

$$X(0)=0, \quad X(1)=0. \tag{11.7}$$

The eigenvalues in this case are computed rather simply.

Indeed, since according to (11.6),

$$X_1(0,\varepsilon)\neq 0, \quad X_2(0,\varepsilon)=0, \tag{11.8}$$

we can satisfy the first condition (11.7) by setting $C_1=0$ in (11.5). Consequently /62

$$Y(\tau,\varepsilon)=X_2(\tau,\varepsilon)C_2. \tag{11.9}$$

Utilizing the second condition from (11.7) we obtain

$$X_2(1,\varepsilon)C_2=0. \tag{11.10}$$

The system of equations (11.10) will have a non-trivial solution only when

$$\det|X_2(1,\varepsilon)| = \left|\Pi_\nu(\tau,\varepsilon)\, e^{\frac{1}{\varepsilon}\int_0^1 D_\nu(\tau,\varepsilon)d\tau} \sin\frac{1}{\varepsilon}\int_0^1 \Omega_\nu(\tau,\varepsilon)\,d\tau\right| = 0, \tag{11.11}$$

$$\nu = 1, 2, \ldots, n,$$

from which it follows that

$$\det|X_2(1,\varepsilon)| = e^{\frac{1}{\varepsilon}\int_0^1 \sum_{\nu=1}^{n} D_\nu(\tau,\varepsilon)d\tau} \prod_{\nu=1}^{n} \sin\frac{1}{\varepsilon}\int_0^1 \Omega_\nu(\tau,\varepsilon)\,d\tau\,|\,\Pi_\nu(\tau,\varepsilon)|_{\nu=1}^{n} = 0. \tag{11.12}$$

It follows from equation (11.12) that

$$\sin\frac{1}{\varepsilon}\int_0^1 \Omega_\nu(\tau,\varepsilon)\,d\tau = 0, \quad \nu = 1, 2, \ldots, n,$$

or

$$\int_0^1 \Omega_\nu(\tau,\varepsilon)\,d\tau = K_\nu\pi\varepsilon, \quad K_\nu = 1, 2, 3, \ldots.$$

Taking into account (7.3), we obtain

$$\int_0^1 [\Omega_{\nu 0}(\tau) + \varepsilon\Omega_{\nu 1}(\tau) + \ldots + \varepsilon^m\Omega_{\nu m}(\tau) + O(\varepsilon^{m+1})]\,d\tau = K_\nu\varepsilon\pi. \tag{11.13}$$

Consequently, in the zero approximation we have

$$\lambda_{\nu,k}^{(0)} = \frac{[\pi K_\nu - O(1)]^2}{\left(\int_0^1 \sqrt{\omega_\nu(\tau)}\,d\tau\right)^2}. \tag{11.14}$$

Assuming that m=1 in (11.13), we find /63

$$\lambda_{\nu,k}^{(1)} = \frac{\left[\pi K_\nu - \int_0^1 \Omega_{\nu 1}(\tau)\,d\tau - O\left(\frac{1}{\sqrt{\lambda}}\right)\right]^2}{\left(\int_0^1 \sqrt{\omega_\nu(\tau)}\,d\tau\right)^2}$$

It is easy to obtain the second and subsequent approximations for the eigenvalues $\lambda_{\nu,k}$.

We shall now illustrate the above by means of examples.

Example 1. It is required to find the eigenvalues of the following boundary value problem:

$$y'' + \lambda(1 + x)y = 0, \quad y(0) = y(1) = 0.$$

We seek the solution in the form

$$y = \xi, \quad \frac{d\xi}{dt} = [D(\tau, \varepsilon) + i\Omega(\tau, \varepsilon)]\,\xi, \quad x = \varepsilon t = \tau, \quad \varepsilon = \frac{1}{\sqrt{\lambda}}.$$

According to the equations of Section 7, we have

$$\Omega_0(\tau) = \pm\sqrt{1 + \tau}, \quad \Omega_1(\tau) = 0, \quad D_1(\tau) = -\frac{1}{8(1 + \tau)},$$

$$\Omega_2(\tau) = \pm\frac{5}{32(1 + \tau)^{9/2}}, \quad D_2(\tau) = 0.$$

From equality (11.13) we have ε=2.553; λ=6.52. The exact solution of this problem is λ=6.55 (compared with example in ref. 26).

Example 2. We must find the eigenvalues of the boundary value problem

$$y^{(IV)} - \lambda(1 + x)y = 0,$$
$$y(0) = y''(0) = 0, \quad y(1) = y''(1) = 0. \tag{11.15}$$

First we must reduce this equation to the form (11.3). We change the variables as follows:

$$y = u_1, \qquad \frac{d^2y}{dx^2} = u_2.$$

Also, in order to obtain symmetric matrices we consider the matrix V(x)=

$\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\lambda(1 + x)} \end{bmatrix}$. Then, if we set $u = V(x)z, \quad x = \varepsilon t = \tau, \quad \varepsilon = \frac{1}{\sqrt[4]{\lambda}}$,
where

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix},$$

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we reduce the problem (11.15) to the form

$$\frac{d^2z}{dt^2} + \varepsilon C(\tau, \varepsilon)\frac{dz}{dt} + B(\tau, \varepsilon)z = 0, \tag{11.16}$$

where

$$C(\tau, \varepsilon) = \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{1}{1+\tau} \end{bmatrix}, \quad B(\tau, \varepsilon) = \begin{bmatrix} 0 & \sqrt{1+\tau} \\ \sqrt{1+\tau} & -\dfrac{\varepsilon^2}{4(1+\tau)^2} \end{bmatrix},$$

$$Z_1(0) = Z_2(0) = 0; \quad Z_1(1) = Z_2(1) = 0.$$

In this case $\Omega_{\nu 0}(\tau) = \pm\sqrt{\omega_\nu(\tau)}$,

$$\omega_\nu(\tau) = \pm\sqrt{1+\tau}, \quad \Omega_1(\tau) = 0, \quad \nu = 1, 2,$$

$$D_1(\tau) = -\frac{1}{8(1+\tau)}, \quad \mu = \pm\frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

It is easy to see that in the first approximation $\lambda=65.8$. This example is taken from reference 27 where in the first approximation $\underline{\lambda}=63.2, \overline{\lambda}=67.2$, the average value $\lambda=65.2$. Here $\underline{\lambda}$ and $\overline{\lambda}$ are respectively the upper and lower approximations of the solution determined by the method of S. A. Chaplygin.

CHAPTER 3. ASYMPTOTIC PARTITION OF THE SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS

Section 12. Formulation of the Problem

In the preceding chapters we have considered the problem of constructing the asymptotic solutions for one differential equation of the second order and of system n of such equations in the case of simple eigenvalues. However, the method which has been presented cannot be applied if among the roots of the characteristic equation we encounter multiple roots over the entire interval [0,L] or at its individual points. /65

In this case the investigation of the system of differential equations of a higher order may be simplified by an asymptotic partitioning of the initial system into several subsystems of lower order. This chapter is concerned with the presentation of this problem.

Let us consider a system of linear differential equations of the form

$$\frac{dx}{dt} = A(\tau, \varepsilon)\, x, \tag{12.1}$$

where $x(t,\varepsilon)$ is an n-dimensional vector, $A(\tau,\varepsilon)$ is a real square matrix of order n, $\tau=\varepsilon t$, and $\varepsilon>0$ is a small parameter.

Let us assume that the following conditions are satisfied:

1) The matrix $A(\tau,\varepsilon)$ can be expanded in the form

$$A(\tau, \varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s A_s(\tau) \tag{12.2}$$

and is infinitely differentiable* with respect to τ over the interval $[0,L]$, where $L>0$ is a fixed number.

In particular, the matrix $A(\tau,\varepsilon)$ may be some polynomial in ε.

2) The roots $\lambda_j(\tau)$ $(j=1, 2, \ldots, n)$ of the characteristic equation of the matrix $A_0(\tau)$

$$D(\lambda) \equiv \det [A_0(\tau) - \lambda(\tau) E] = 0, \tag{12.3}$$

if $\tau\in[0,L]$ form two isolated groups, i.e., groups such that $\lambda_i(\tau) \neq \lambda_j(\tau)$, where /66

$i = 1, 2, \ldots, r_1;\ j = r_1+1, r_1+2, \ldots, n;\ n - r_1 = r_2$. Within each group the roots may have an arbitrary multiplicity (which is different for different τ).

*A matrix is considered differentiable if all its elements are differentiable.

We shall show later that, under the assumptions which have been made, the system considered (12.1) may be partitioned asymptotically into two independent systems and the sum of the orders of these is equal to the order of the given system.

When we construct the algorithm for the partition we shall require a non-singular matrix $V(\tau)$ which will transform $A_0(\tau)$ into a quasi-diagonal form

$$W_0(\tau) \equiv V^{-1}(\tau) A_0(\tau) V(\tau) = \begin{bmatrix} W_{10}(\tau) & 0 \\ 0 & W_{20}(\tau) \end{bmatrix}, \qquad (12.4)$$

where $W_{10}(\tau)$ is a square matrix of order r_1 with eigenvalues of the first group $\lambda_i(\tau)$, and $W_{20}(\tau)$ is a square matrix of the r_2-th order with eigenvalues of the second group $\lambda_j(\tau)$ $(i = 1, 2, \ldots, r_1;\ j = r_1 + 1,\ r_1 + 2, \ldots, n;\ r_1 + r_2 = n)$.

As we know, when condition 2 is satisfied, such a transformation matrix $V(\tau)$ may be specified for each $\tau \in [0, L]$. The construction and properties of matrix $V(\tau)$ will be considered in detail in Section 14.

Section 13. Formal Partition

The possible formal partition of a system of linear differential equations (12.1) into two independent subsystems is established by the following theorem.

Theorem 3.1. If the matrix $A(\tau, \varepsilon)$ satisfies conditions 1 and 2 of Section 12, the formal solution of system (12.1) may be represented in the form

$$x(t, \varepsilon) = U(\tau, \varepsilon) \xi(t, \varepsilon), \qquad (13.1)$$

where $U(\tau, \varepsilon)$ is a square matrix of order n, while the n-dimensional vector $\xi(t, \varepsilon)$ satisfies the equation

$$\frac{d\xi}{dt} = \mathfrak{A}(\tau, \varepsilon) \xi, \qquad (13.2)$$

in which the matrix $\mathfrak{A}(\tau, \varepsilon)$ has a quasi-diagonal structure

$$\mathfrak{A}(\tau, \varepsilon) = \begin{bmatrix} \mathfrak{A}_1(\tau, \varepsilon) & 0 \\ 0 & \mathfrak{A}_2(\tau, \varepsilon) \end{bmatrix}, \qquad (13.3)$$

and $\mathfrak{A}_k(\tau, \varepsilon)$ are square matrices of order r_k (k=1, 2).

It is assumed that matrices $U(\tau, \varepsilon)$, $\mathfrak{A}(\tau, \varepsilon)$ (and consequently $\mathfrak{A}_k(\tau, \varepsilon)$) admit formal expansion in the form /67

$$U(\tau, \varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s U_s(\tau), \tag{13.4}$$

$$\mathfrak{A}(\tau, \varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s \mathfrak{A}_s(\tau) \quad \text{or} \quad \mathfrak{A}_k(\tau, \varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s \mathfrak{A}_{ks}(\tau). \tag{13.5}$$

Thus, according to the structure of matrix $\mathfrak{A}(\tau, \varepsilon)$, equation (13.2) may be looked upon as a system consisting of two independent subsystems, where

$$\xi(t, \varepsilon) = \begin{bmatrix} \xi_1(t, \varepsilon) \\ \xi_2(t, \varepsilon) \end{bmatrix} \text{ and } \frac{d\xi_1}{dt} = \mathfrak{A}_1(\tau, \varepsilon)\xi_1, \quad \frac{d\xi_2}{dt} = \mathfrak{A}_2(\tau, \varepsilon)\xi_2$$

(the orders of the systems obtained by partition are r_1 and r_2 respectively).

Proof of Theorem 3.1 will consist of the construction of an algorithm which can be used to determine the terms of the expansion (13.4) and (13.5). For this purpose, taking into account (13.2), we substitute expression (13.1) into system (12.1). In the resulting identity we equate the coefficients of the vector function $\xi(t, \varepsilon)$ and obtain a matrix differential equation

$$\varepsilon \frac{dU}{d\tau} + U(\tau, \varepsilon)\mathfrak{A}(\tau, \varepsilon) = A(\tau, \varepsilon)U(\tau, \varepsilon), \tag{13.6}$$

which must be satisfied by the unknown matrices $U(\tau, \varepsilon)$ and $\mathfrak{A}(\tau, \varepsilon)$.

If in relationship (13.6) we successively isolate the terms in front of the same powers of parameter ε, we shall obtain an infinite system of algebraic matrix equations for determining the unknown elements of expansions (13.4) and (13.5):

$$\begin{gathered} A_0(\tau)U_0(\tau) - U_0(\tau)\mathfrak{A}_0(\tau) = 0, \\ A_0(\tau)U_1(\tau) - U_1(\tau)\mathfrak{A}_0(\tau) = U_0(\tau)\mathfrak{A}_1(\tau) - A_1(\tau)U_0(\tau) + \\ + \frac{dU_0}{d\tau} \equiv U_0(\tau)\mathfrak{A}_1(\tau) + B_1(\tau), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ A_0(\tau)U_s(\tau) - U_s(\tau)\mathfrak{A}_0(\tau) = U_0(\tau)\mathfrak{A}_s(\tau) + B_s(\tau), \end{gathered} \tag{13.7}$$

where

$$B_s(\tau) = \sum_{j=1}^{s-1} U_j(\tau)\mathfrak{A}_{s-j}(\tau) - \sum_{j=1}^{s} A_j(\tau)U_{s-j}(\tau) + \frac{dU_{s-1}}{d\tau} \tag{13.8}$$

(The matrix $B_s(\tau)$ contains the unknown matrices with numbers which are less than s, and the derivatives with respect to τ of matrix $U_{s-1}(\tau)$.)

We shall assume for the present that all the necessary derivatives of the elements of $U_s(\tau)$ with respect to τ exist. In Section 14 we shall return to the question of proving the differentiability of matrices $U_s(\tau)$. /68

We multiply both parts of the left side of each equation in system (13.6) by the matrix $V^{-1}(\tau)$ (see Section 12) and by introducing a new unknown matrix

$$Q(\tau) = V^{-1}(\tau) U_s(\tau), \tag{13.9}$$

we obtain, in accordance with (13.7), the following system of equations:

$$\begin{gathered} W_0(\tau) Q_0(\tau) - Q_0(\tau) \mathfrak{A}_0(\tau) = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ W_0(\tau) Q_s(\tau) - Q_s(\tau) \mathfrak{A}_0(\tau) = Q_0(\tau) \mathfrak{A}_s(\tau) + F_s(\tau), \end{gathered} \tag{13.10}$$

where

$$F_s(\tau) = V^{-1}(\tau) B_s(\tau), \qquad s = 1, 2, 3, \ldots . \tag{13.11}$$

Proceeding to the solution of system (13.10), we consider the first equation.

We let

$$Q_0(\tau) = E, \tag{13.12}$$

where E is a unit matrix of order n.

Then, according to the first equation of system (13.10),

$$\mathfrak{A}_0(\tau) = W_0(\tau), \tag{13.13}$$

and the last equation of (13.10) may be rewritten in the form

$$\begin{gathered} W_0(\tau) Q_s(\tau) - Q_s(\tau) W_0(\tau) = \mathfrak{A}_s(\tau) + F_s(\tau), \\ s = 1; 2, \ldots . \end{gathered} \tag{13.14}$$

Having determined the zero terms of the matrix $U(\tau, \varepsilon)$ expansion in accordance with (13.9) and (13.12)

$$U_0(\tau) = V(\tau), \tag{13.15}$$

we can proceed with the solution of equations (13.14) when $s \geq 1$. We note that the matrix $F_s(\tau)$, according to (13.8) and (13.11), contains only those terms of expansions (13.4) and (13.5) whose numbers are less than s. Therefore,

during the successive solution (s=1, 2, ...) of equation (13.14) we must consider the matrix $F_s(\tau)$ to be known.

To construct the solution of equation (13.14) we partition the matrices $Q_s(\tau)$ and $F_s(\tau)$ into cells corresponding to the quasi-diagonal structure of the known matrix $W_0(\tau)$ and of the unknown matrix $\mathfrak{U}(\tau, \varepsilon)$ (see (12.4) and (13.3)):

$$Q_s(\tau) = \begin{bmatrix} Q_{11s}(\tau) & Q_{12s}(\tau) \\ Q_{21s}(\tau) & Q_{22s}(\tau) \end{bmatrix} \qquad F_s(\tau) = \begin{bmatrix} F_{11s}(\tau) & F_{12s}(\tau) \\ F_{21s}(\tau) & F_{22s}(\tau) \end{bmatrix} \tag{13.16}$$

where $Q_{ijs}(\tau)$ and $F_{ijs}(\tau)$ are matrices of the order (r_i, r_j) (i=1, 2; j=1, 2). Then equation (13.14) may be represented in the form of the following equivalent system of matrix equations: /69

$$W_{i0}(\tau) Q_{ijs}(\tau) - Q_{ijs}(\tau) W_{j0}(\tau) = \delta_{ij} \mathfrak{A}_{is}(\tau) + F_{ijs}(\tau), \tag{13.17}$$

where δ_{ij} is the Kronecker symbol; i, j=1, 2. Let us find the solution of this system.

We let i=j and

$$Q_{iis} = E_{r_i} \tag{13.18}$$

(E_{ri} is a unit matrix of order r_i).

In this case, according to (13.17), we find

$$\mathfrak{A}_{is}(\tau) = -F_{iis}(\tau) \qquad (i = 1, 2). \tag{13.19}$$

When i≠j, equation (13.17) may be represented in the form

$$W_{i0}(\tau) Q_{ijs}(\tau) - Q_{ijs}(\tau) W_{j0}(\tau) = F_{ijs}(\tau). \tag{13.20}$$

The unknown matrix $Q_{ijs}(\tau)$ (i≠j) is determined from equation (13.20) uniquely because the corresponding homogeneous equation

$$W_{i0}(\tau) Q_{ijs}(\tau) - Q_{ijs}(\tau) W_{j0}(\tau) = 0 \tag{13.21}$$

has only a zero solution.

The last proposition follows from the fact that the matrices $W_{i0}(\tau)$ and $W_{j0}(\tau)$ (i≠j) for any $\tau \in [0, L]$, according to proposition 2 of Section 12, do not have equal eigenvalues (see Chapter 8 of reference 15).

Thus, having determined the cells of matrix $Q_s(\tau)$ from system (13.17), we can reconstruct the matrix $Q_s(\tau)$ itself and consequently determine the corresponding expansion term of matrix $U(\tau, \varepsilon)$:

$$U_s(\tau) = V(\tau) Q_s(\tau). \tag{13.22}$$

Thus, the method of solution described here shows how we can find the elements of formal expansions (13.4) and (13.5), i.e., the matrices $U_s(\tau)$ and $\mathfrak{U}_s(\tau)$ with any number s=0, 1, 2, ...

To complete the description of the algorithm for the formal partitioning of system (12.1) we must show that when the conditions of Theorem 3.1 are satisfied the matrices $U_s(\tau)$ can be differentiated with respect to τ a sufficient number of times. This question will be considered in detail in the next section.

Section 14. Construction of the Transformation Matrix and the Differentiability of the Formal Solution

Before we investigate the differentiability of the matrices $U_s(\tau)$(s=0, 1, 2, ...) with respect to τ, we prove the following lemma. /70

Lemma 3.1. If in the segment $[0, L]$ the matrix $A_0(\tau)$ has a certain number of derivatives with respect to τ, then the nonsingular transformation matrix $V(\tau)$ can be selected in such a way that it, and consequently the quasi-diagonal matrix $W_0(\tau)$ (see (12.4)), will have the same number of derivatives in the segment $[0, L]$.

Proof. To construct the transformation matrix $V(\tau)$ we shall decompose all of our space R_n into a direct sum of two subspaces which are invariant with respect to the matrix $A_0(\tau)$ and which correspond to two groups of eigenvalues. As we know (ref. 68), the specified expansion may be achieved by means of matrices $P_1(A_0)$ and $P_2(A_0)$, which are projection matrices in the sense that

$$P_k^2(A_0) = P_k(A_0) \quad (k = 1, 2). \tag{14.1'}$$

These matrices have the following properties:

$$\begin{gathered} P_1(A_0) + P_2(A_0) = E, \qquad A_0 P_k = P_k A_0 \qquad (k = 1, 2), \\ P_1(A_0) P_2(A_0) = P_2(A_0) P_1(A_0) = 0. \end{gathered} \tag{14.1"}$$

Let us recall the method of constructing the matrices $P_k(A_0)$.

In accordance with proposition 2 of Section 12, we represent the characteristic polynomial $D(\lambda)$ of the matrix $A_0(\tau)$ in the form of a product of two mutually prime cofactors, each of which corresponds to a definite group of eigenvalues

$$D(\lambda) = D_1(\lambda) D_2(\lambda), \tag{14.2}$$

where $D_1(\lambda)$ is a polynomial of the r_1-th degree and $D_2(\lambda)$ is a polynomial of the r_2-th degree ($r_1+r_2=n$).

According to (14.2) the following equation exists:

$$\frac{1}{D(\lambda)} = \frac{d_1(\lambda)}{D_1(\lambda)} + \frac{d_2(\lambda)}{D_2(\lambda)},$$

and consequently the following identity is valid:

$$E = d_1(A_0) D_2(A_0) + d_2(A_0) D_1(A_0). \tag{14.3}$$

Here $d_k(\lambda)$ is a polynomial of a degree not greater than r_k-1 which is mutually prime with $D_k(\lambda)$ (k=1, 2).

We let

$$\begin{aligned} P_1(A_0) &= d_1(A_0) D_2(A_0), \\ P_2(A_0) &= d_2(A_0) D_1(A_0). \end{aligned} \tag{14.4}$$

Utilizing relationship (14.3) and the Cayley identity $D(A_0)=0$, it is easy to see that equalities (14.1') and (14.1") are valid, i.e., the matrices $P_1(A_0)$ and $P_2(A_0)$ indeed spread out the space R_n into a complete system of subspaces P_1R_n and P_2R_n which are invariant with respect to $A_0(\tau)$ such that the sum of their measurements is equal to n. /71

We shall now show that subspace P_kR_n, which is determined by the formula

$$x_k = P_k x, \qquad k = 1, 2, \tag{14.5}$$

may be assigned by an equation of the form

$$D_k(A_0) x_k = 0, \qquad k = 1, 2, \tag{14.6}$$

i.e., the subspace P_kR_n is a set of vectors which satisfy equation (14.6).

Indeed, let us first assume that there is a vector x_k which is determined by equation (14.5). Then by substituting expression (14.5) in place of x_k into equation (14.6), we obtain

$$D_k(A_0)P_kx = d_k(A_0)D(A_0)x.$$

However $D(A_0)=0$ and consequently the vectors (14.5) satisfy equation (14.6).

The converse is also true: the solution of equation (14.6) belongs to the subspace (14.5).

Indeed according to (14.1), each vector of the space R_n, including any solution η_1 of equation $D_1(A_0)\eta_1=0$, may be represented in the form

$$\eta_1 = (P_1 + P_2)\eta_1 \equiv d_1(A_0)D_2(A_0)\eta_1 + d_2(A_0)D_1(A_0)\eta_1. \qquad (14.7)$$

However $D_1(A_0)\eta_1=0$; consequently

$$\eta_1 = d_1(A_0)D_2(A_0)\eta_1 \equiv P_1(A_0)\eta_1,$$

i.e., the solution of equation (14.6) when k=1 belongs to the subspace P_1R_n.

The same proposition obtains in the case when k=2.

We know that the set of solutions for equation $D_k(A_0)x=0$(k=1, 2) is not an empty subspace, as we can easily see if we write equation (14.6) in the form

$$D_1(A_0)x = \prod_{i=1}^{r_1}(A_0 - \lambda_i E)x = 0,$$

$$D_2(A_0)x = \prod_{j=r_1+1}^{n}(A_0 - \lambda_j E)x = 0.$$

We let the dimension of subspaces P_1R_n and P_2R_n be equal to α_1 and α_2 ($\alpha_1+\alpha_2=n$) /72 respectively. As the new basic orthogonal unit vectors for our n-dimensional space α_1 of base vectors, we select the subspaces P_1R_n and for space α_2 of base vectors we select the subspaces P_2R_n.

In accordance with the invariance of subspaces P_kR_n with respect to $A_0(\tau)$

the matrix $A_0(\tau)$ in this new base will have a quasi-diagonal form and the specified transformation is achieved by means of the matrix $T(\tau)$:

$$T^{-1}(\tau) A_0(\tau) T(\tau) = \Lambda(\tau),$$

where $T(\tau)$ is a square matrix of the n-th order, composed of the new base vectors;

$$\Lambda(\tau) = \begin{bmatrix} \Lambda_1(\tau) & 0 \cdot \\ 0 & \Lambda_2(\tau) \end{bmatrix}; \qquad (14.8)$$

$\Lambda_1(\tau)$ is a matrix of order α_1; $\Lambda_2(\tau)$ is a matrix of order α_2.

We shall show that $\alpha_1=r_1, \alpha_2=r_2$ and that the eigenvalues $\mu_m (m=1, 2, \ldots, \alpha_1)$ of the matrix $\Lambda_1(\tau)$ are the eigenvalues of the first group $\lambda_i(\tau)$ $(i=1, 2, \ldots, r_1)$, while the eigenvalues $\mu_t (t=\alpha_1 + 1, \alpha_1 + 2, \ldots, n)$ of the matrix $\Lambda_2(\tau)$ belong to the second group $\lambda_j(\tau)$ $(j=r_1 + 1, r_2 + 2, \ldots, n)$ of roots of the characteristic polynomial of the matrix $A_0(\tau)$.

Let us take any vector y from the subspace $P_1 R_n$. It satisfies equation (14.6) when k=1 and consequently (in the new coordinates) it satisfies equation

$$T^{-1}(\tau) D_1(A_0) T(\tau) y = 0,$$

which apparently may be rewritten in the form

$$D_1(T^{-1} A_0 T) y = 0. \qquad (14.9)$$

Since $y \in P_1 R_n$, its last components $\alpha_2=n-\alpha_1$ are equal to zero; consequently, equation (14.9) is equivalent to the equation

$$D_1(\Lambda_1) y_1 = 0, \qquad (14.10)$$

where y_1 is any α_1-dimensional vector.

Since equality (14.10) exists for any $y_1, D_1(\Lambda_1) \equiv 0$, and consequently all the characteristic numbers of matrix $D_1(\Lambda_1)$ also equal zero. However, all these characteristic numbers are formed from the characteristic numbers of matrix Λ_1 by means of equation $D_1(\mu_m)$. Consequently any eigenvalue of the matrix Λ_1 is a root of the equation

$$D_1(\mu_m) = 0 \qquad (m = 1, 2, \ldots, \alpha_1).$$

By carrying out similar arguments we can become convinced that any eigenvalue of the matrix Λ_2 is a root of equation

$$D_2(\mu_l) = 0 \qquad (l = \alpha_1 + 1, \alpha_1 + 2, \ldots, n).$$

Since such matrices $A_0(\tau)$ and $\Lambda(\tau)$ have the same characteristic numbers and since $D_1(\lambda)$ and $D_2(\lambda)$ are mutually prime polynomials, it follows that $\alpha_1=r_1$ and $\alpha_2=r_2$ and that the eigenvalues of the matrix $\Lambda_k(\tau)$ coincide with the roots of the corresponding polynomials $D_k(\lambda)(k=1, 2)$.

Thus we may select the matrix $T(\tau)$, which we have constructed, as the unknown transformation matrix.

It remains to be shown that this matrix has the same number of derivatives with respect to τ as $A_0(\tau)$.

To do this we return to the problem of constructing the base vectors which form the matrix T. As follows from the arguments which have been presented, the vectors belonging to subspaces P_1R_n and P_2R_n are solutions of the system of algebraic equations

$$P_2(A_0(\tau))\,x_1 = d_2(A_0(\tau))\,D_1(A_0(\tau))\,x_1 = 0 \qquad (14.11)$$

and

$$P_1(A_0(\tau))\,x_2 = d_1(A_0(\tau))\,D_2(A_0(\tau))\,x_2 = 0 \qquad (14.12)$$

respectively.

Therefore we must first become convinced that the matrices $P_k(A_0(\tau))(k=1, 2)$ have the same number of derivatives with respect to τ as the matrix $A_0(\tau)$. Apparently, in this connection it is sufficient to show that the coefficients of the polynomial $P_k(\lambda)$ are differentiable with respect to τ at least the same number of times as the elements of the matrix $A_0(\tau)$.

Let us prove the last proposition.

As we have already stated, according to the expansion

$$D(\lambda) = D_1(\lambda)\,D_2(\lambda), \qquad (14.13)$$

the following equation exists:

$$\frac{1}{D(\lambda)} = \frac{d_1(\lambda)}{D_1(\lambda)} + \frac{d_2(\lambda)}{D_2(\lambda)}, \qquad (14.14)$$

where $d_k(\lambda)$ is a polynomial of power not greater than r_k-1 and mutually prime with $D_k(\lambda)(k=1, 2)$.

The roots of the polynomials $D_1(\lambda)$ and $D_2(\lambda)$ form two isolated groups in the complex plane. Therefore we can construct two smooth nonintersecting contours Γ_1 and Γ_2 in this plane such that these contours contain the roots of the respective polynomials.

Since the function $\frac{d_1(\lambda)}{D_1(\lambda)}$ is holomorphic in the closed domain bounded by /74 the contour Γ_2, the Cauchy equation is valid:

$$\frac{d_1(\lambda)}{D_1(\lambda)} = \frac{1}{2\pi i}\int_{\Gamma_2} \frac{d_1(\mu)\,d\mu}{D_1(\mu)(\mu-\lambda)}, \qquad (14.15)$$

where λ is any point inside the contour Γ_2 which is integrated in the positive direction.

The function $\frac{d_2(\lambda)}{D_2(\lambda)}$ is holomorphic outside the region bounded by Γ_2, and when $\lambda \to \infty$ it tends uniformly to zero.

Therefore

$$0 = \frac{1}{2\pi i}\int_{\Gamma_2} \frac{d_2(\mu)\,d\mu}{D_2(\mu)(\mu-\lambda)}, \qquad (14.16)$$

where λ is a point inside the contour Γ_2 and the contour is traced in the same direction as in equation (14.15).

Adding equalities (14.15) and (14.16), we obtain, in accordance with (14.14),

$$\frac{d_1(\lambda)}{D_1(\lambda)} = \frac{1}{2\pi i}\int_{\Gamma_2} \frac{d\mu}{D(\mu)(\mu-\lambda)}. \qquad (14.17)$$

On the basis of equation (14.17) we can maintain that the function $\frac{d_1(\lambda)}{D_1(\lambda)}$ is differentiable with respect to the parameter τ the same number of times as the characteristic polynomial $D(\lambda,\tau)$ of the matrix $A_0(\tau)$, i.e., the same number of times as the elements of the matrix $A_0(\tau)$; the coefficients of the polynomial $D(\lambda,\tau)$ are multinomials formed from the elements of the matrix $A_0(\tau)$. Thus it is quite obvious that the function $P_1(\lambda,\tau)$ can be differentiated with respect to τ, because

$$P_1(\lambda) = \frac{d_1(\lambda)}{D_1(\lambda)} \cdot D(\lambda) = d_1(\lambda) D_2(\lambda),$$

and each of the functions $\frac{d_1(\lambda)}{D_1(\lambda)}$ and $D(\lambda)$ can be differentiated with respect to τ the same number of times as the matrix $A_0(\tau)$.

A similar assertion is valid for the polynomial

$$P_2(\lambda) = d_2(\lambda) D_1(\lambda).$$

Now let us prove that the solutions of equations (14.11) and (14.12) are differentiable with respect to τ.

In view of our preceding arguments, the system of n algebraic equations (14.11) has r_1 independent solutions, i.e., the rank of matrix $P_2(A_0(\tau))$ is /75 equal to r_2. Therefore if in equation (14.11) we isolate a minor different from zero of the r_2-th order and assign corresponding values to the r_1 arbitrary components of the vector x_1 (for example set them equal to constants), we can differentiate equation (14.11) with respect to it and be convinced that the solution $x_1(\tau)$ is differentiable the same number of times as the matrix $P_2(A_0(\tau))$, i.e., as the elements of the matrix $A_0(\tau)$. The same statements can be made relative to the differentiability of r_2 independent solutions of system (14.12).

Thus, when we construct the transformation matrix we can always find spaces of differentiable solutions of equations (14.11) and (14.12) which have the same number of derivatives as the matrix $A_0(\tau)$.

If we take, as the new base, the orthonormalized systems of independent differentiable solutions of equations (14.11) and (14.12), (which as we know is always possible), the transformation matrix $V(\tau)=[V_1,V_2]$ constructed from them will have all of the properties specified in Lemma 3.1. In addition to this the following equality exists

$$V_k^*(\tau)\,V_k(\tau)=E_{r_k},\qquad k=1,2, \tag{14.18}$$

where $V_k^*(\tau)$ is a matrix which has been transposed to the matrix $V_k(\tau)$. It is obvious that the matrix $W_0(\tau)=V^{-1}(\tau)A_0(\tau)V(\tau)$ will also have the same number of derivatives as $V(\tau)$, i.e., the same numbers as $A_0(\tau)$. Lemma 3.1 has been proved.

Now by using the statements of Lemma 3.1, we shall prove the differentiability of matrices $U_s(\tau)$(s=1, 2, 3, ...).

Since, according to (13.22),

$$U_s(\tau)=V(\tau)Q_s(\tau),$$

the question of the differentiability of elements $U_s(\tau)$ may be solved after investigating the properties of the matrix $Q_s(\tau)$(s=1, 2, ...).

The matrix $Q_s(\tau)$ will be differentiable if all of its cells $Q_{ijs}(\tau)$(i=1, 2; j=1, 2; s=1, 2, ...) are differentiable. The differentiability of cells $Q_{ijs}(\tau)$ follows immediately from equality (13.18). When $i\neq j$ the matrices $Q_{ijs}(\tau)$ represent the only solution of equation (13.20). By carrying out differentiation with respect to τ in (13.20) we arrive at the conclusion that $Q_{ijs}(\tau)$ has the same number of derivatives as the matrices $W_{i0}(\tau)$(i=1, 2), i.e., as the matrix $W_0(\tau)$ and $F_{ijs}(\tau)$. We recall that $F_{ijs}(\tau)$ are the cells of the matrix $F_s(\tau)$ which is formed from the expansion terms (12.2) of matrix $A(\tau,\varepsilon)$, matrices $\mathfrak{U}_m(\tau)$, $U_m(\tau)$($m\leq s$), and the derivatives of the latter.

Thus if we assume that the elements $A_s(\tau)$ have a sufficient number of derivatives, i.e., that condition 1 of Section 12 is satisfied, then $F_s(\tau)$ and consequently $Q_{ijs}(\tau)$ will have the required number of derivatives. /76

This statement shows that all of the derivatives of matrices $U_s(\tau)$(s=0, 1, 2, ...) with respect to τ, which we must use to operate in the process of a formal partitioning of system (12.1), do indeed exist under propositions which have been made.

The establishment of differentiability for matrices $U_s(\tau)$ concludes the proof of Theorem 3.1.

Section 15. Proof of Asymptotic Convergence

In Section 13 we presented the algorithm for the formal partitioning of the system of differential equations (12.1) into two linearly independent systems.

In the present section we shall show that the formal solution (13.1) constructed in accordance with the specified algorithm converges asymptotically to the exact solution $x(t,\varepsilon)$ of system (12.1), i.e., the norm* of the difference between the exact solution and the approximation of the m-th order tends to zero when m is fixed and when $\varepsilon \to 0$.

As before, we shall interpret the m-th order approximation as the vector $x^{(m)}(t,\varepsilon)$ of the form

$$x^{(m)}(t,\varepsilon) = U^{(m)}(\tau,\varepsilon)\,\xi^{(m)}(t,\varepsilon), \tag{15.1}$$

where $\xi^{(m)}(t,\varepsilon)$ is determined by equation (13.2) if in the formal expansion (13.4), (13.5) we limit ourselves to the first m+1 terms.

In obtaining an estimate of the difference

$$z(t,\varepsilon) = x(t,\varepsilon) - x^{(m)}(t,\varepsilon) \tag{15.2}$$

we shall require the following lemma.

Lemma 3.2. If the matrix $A(\tau,\varepsilon) = A_0(\tau) + \varepsilon\bar{A}(\tau,\varepsilon)$, where $\bar{A}(\tau,\varepsilon) = \sum_{s=1}^{\infty}\varepsilon^{s-1}A_s(\tau)$ satisfies the conditions of Theorem 3.1, and if, in addition, for any n-dimensional vector y the following inequality is valid:

$$\mathrm{Re}\,(A_0(\tau)\,y, y) \leqslant 0, \tag{15.3}$$

which is equivalent to the nonpositive nature of the eigenvalues of the symmetrical matrix $\frac{A_0+A_0^*}{2}$, then for the solution of equation /77

$$\frac{dx}{dt} = [A_0(\tau) + \varepsilon\bar{A}(\tau,\varepsilon)]\,x \tag{15.4}$$

with initial condition $x(0,\varepsilon)=x_0$ the following estimate is valid on the segment $[0, L/\varepsilon]$ $0<\varepsilon\leq\varepsilon_0$,

*

We have in mind the norms of conventional unitary space:

$$\|y\| = \sqrt{(y,y)}, \qquad \|A\| = \max_i \sum_{j=1}^{n} |a_{ij}|.$$

$$\| x(t, \varepsilon) \| \leqslant C \| x_0 \|, \tag{15.5}$$

where C is some positive constant which is independent of ε.

Proof. Let $\Psi(t, \varepsilon)=(x(t, \varepsilon))$, where $x(t, \varepsilon)$ is the solution of system (15.4), and $x(t, \varepsilon)\big|_{t=0}=x_0$. Then, in view of (15.3), we have

$$\frac{d\Psi}{dt} = (A(\tau, \varepsilon) x, x) + (x, A(\tau, \varepsilon) x) = 2\mathrm{Re}\,(A(\tau, \varepsilon) x, x) = \tag{15.6}$$

$$= 2\mathrm{Re}\,(A_0(\tau) x, x) + 2\varepsilon\,\mathrm{Re}\,(\bar{A}(\tau, \varepsilon) x, x) \leqslant$$

$$\leqslant 2\varepsilon \sup_{\substack{0 \leqslant \tau \leqslant L, \\ 0 \leqslant \varepsilon \leqslant \varepsilon_0}} \| \bar{A}(\tau, \varepsilon) \| \Psi(t, \varepsilon);$$

or

$$\Psi(t, \varepsilon) \leqslant \Psi(0, \varepsilon) + 2\varepsilon \sup_{\substack{0 \leqslant \tau \leqslant L, \\ 0 < \varepsilon \leqslant \varepsilon_0}} \| \bar{A}(\tau, \varepsilon) \| \int_0^t \Psi(t, \varepsilon)\, dt. \tag{15.7}$$

In accordance with Lemma 1.1, we obtain the following estimate from inequality (15.7):

$$\Psi(t, \varepsilon) \leqslant \Psi(0, \varepsilon)\, e^{2L \sup_{0 \leqslant \tau \leqslant L,\ 0 < \varepsilon \leqslant \varepsilon_0} \| \bar{A}(\tau, \varepsilon) \|},$$

which we rewrite in the form

$$\| x(t, \varepsilon) \| \leqslant C \| x_0 \| \quad \text{for} \quad C = e^{L \sup_{0 \leqslant \tau \leqslant L,\ 0 < \varepsilon \leqslant \varepsilon_0} \| \bar{A}(\tau, \varepsilon) \|},$$

and thus complete the proof of our lemma.

On the basis of Lemma 3.2 we can maintain that for the function $\xi_k^{(m)}(t, \varepsilon)$, k=1, 2, there is also an estimate of form (15.5), because the system

$$\frac{d\xi_k^{(m)}}{dt} = \mathfrak{A}_k^{(m)}(\tau, \varepsilon)\, \xi^{(m)}, \quad k = 1, 2, \tag{15.8}$$

where

$$\mathfrak{A}_k^{(m)}(\tau, \varepsilon) = \mathfrak{A}_{k0}(\tau) + \varepsilon \sum_{s=0}^{m-1} \varepsilon^s \mathfrak{A}_{ks+1}(\tau)$$

satisfies the conditions of Lemma 3.2.

Indeed, according to the determination of matrix $\mathfrak{A}_0(\tau)$ (13.13) /78

and the properties of the transforming matrix $V(\tau)=[V_1(\tau), V_2(\tau)]$, we can write

$$A_0(\tau)V_k(\tau) = V_k(\tau)\mathfrak{A}_{k0}(\tau),$$

or

$$\mathfrak{A}_{k0}(\tau) = V_k^*(\tau)A_0(\tau)V_k(\tau).$$

From this it is obvious that the following inequality is valid:

$$\operatorname{Re}(\mathfrak{A}_{k0}(\tau)y_k, y_k) = \operatorname{Re}(V_k^*(\tau)A_0V_k(\tau)y_k, y_k) = \operatorname{Re}(A_0V_k(\tau)y_k, V_k(\tau)y_k) \leqslant 0,$$

where y_k is an arbitrary vector with dimensionality r_k.

Let us once more consider the difference (15.2).

According to the determination of matrices $U_s(\tau), \mathfrak{A}_s(\tau)$ (s=0, 1, 2, ..., m) (see (13.7)), the vector $x^{(m)}(t,\varepsilon)$ satisfies the equation

$$\frac{dx^{(m)}}{dt} = A^{(m)}(\tau,\varepsilon)x^{(m)} + \varepsilon^{m+1}f(t,\varepsilon), \tag{15.9}$$

where

$$A^{(m)}(\tau,\varepsilon) = \sum_{s=0}^{m}\varepsilon^s A_s(\tau),$$
$$f(t,\varepsilon) = \Phi(\tau,\varepsilon)\xi^{(m)}(t,\varepsilon),$$

and

$$\Phi(\tau,\varepsilon) = \frac{dU_m}{d\tau} + \sum_{s=0}^{m-1}\varepsilon^s\sum_{p=s+1}^{m}[U_p(\tau)\mathfrak{A}_{m+s+1-p}(\tau) - A_{m+s+1-p}(\tau)U_p(\tau)].$$

Consequently, the vector $z(t,\varepsilon)$ is a solution of the system of equations

$$\frac{dz(t,\varepsilon)}{dt} = A(\tau,\varepsilon)z + \varepsilon^{m+1}F_1(t,\varepsilon), \tag{15.10}$$

where

$$F_1(t,\varepsilon) = \widetilde{A}(\tau,\varepsilon)x^{(m)}(t,\varepsilon) - f(t,\varepsilon), \quad \widetilde{A}(\tau,\varepsilon) = \sum_{s=m+1}^{\infty}\varepsilon^{s-m-1}A_s(\tau).$$

The propositions of Section 12, of Lemma 3.2 and of (15.1) show that the vector $F_1(t,\varepsilon)$ is bounded over $[0,\frac{L}{\varepsilon}]$ and that the following estimate is valid for this vector:

$$\| F_1(t, \varepsilon) \| \leqslant \| \widetilde{A}(\tau, \varepsilon) \| \| x^{(m)}(t, \varepsilon) \| + \| f(t, \varepsilon) \| \leqslant$$
$$\leqslant C [\| \widetilde{A}(\tau, \varepsilon) \| \| U^{(m)}(\tau, \varepsilon) \| + \| \Phi(\tau, \varepsilon) \|] \| \xi_0^{(m)} \|,$$

where C is some positive constant and $\xi_0^{(m)}$ is the initial value of the vector $\xi^{(m)}(t, \varepsilon)$.

As we know (ref. 1), the solution of equation (15.10), which satisfies the initial condition $z(t, \varepsilon)|_{t=0} = z_0$, is determined by the equation /79

$$z(t, \varepsilon) = Y(t, \varepsilon) z_0 + \varepsilon^{m+1} \int_0^t Y(t, \varepsilon) Y^{-1}(t_1, \varepsilon) F(t_1, \varepsilon) dt_1, \qquad (15.11)$$

where $Y(t, \varepsilon)$ is the solution of the problem

$$\frac{dY}{dt} = A(\tau, \varepsilon) Y, \quad Y(0, \varepsilon) = E. \qquad (15.12)$$

Indeed, let us assume that

$$z(t, \varepsilon) = Y(t, \varepsilon) u(t, \varepsilon). \qquad (15.13)$$

By substituting expression (15.13) into equation (15.10) we obtain, in accordance with (15.12),

$$\frac{dz}{dt} = A(\tau, \varepsilon) z + Y(t, \varepsilon) \frac{du}{dt} = A(\tau, \varepsilon) z + \varepsilon^{m+1} F(t, \varepsilon),$$

from which it follows that

$$Y(t, \varepsilon) \frac{du}{dt} = \varepsilon^{m+1} F(t, \varepsilon)$$

or

$$u(t, \varepsilon) = u(0, \varepsilon) + \varepsilon^{m+1} \int_0^t Y^{-1}(t_1, \varepsilon) F(t_1, \varepsilon) dt_1. \qquad (15.14)$$

Since $Y(0, \varepsilon) = E$ and according to (15.13), $z_0 = \mu(0, \varepsilon)$, the relationships (15.13) and (15.14) lead to the unknown expression for the solution $z(t, \varepsilon)$ (see (15.11)).

Thus, according to (15.11) the following estimate exists for the vector $z(t, \varepsilon)$:

$$\| z(t, \varepsilon) \| \leqslant \| Y(t, \varepsilon) \| \| z_0 \| + \varepsilon^{m+1} \int_0^t \Psi(t, t_1, \varepsilon) dt_1,$$

where

$$\Psi(t, t_1, \varepsilon) = \| Y(t, \varepsilon) Y^{-1}(t_1, \varepsilon) F(t_1, \varepsilon) \|, \tag{15.15}$$

or

$$\| z(t, \varepsilon) \| \leqslant \| Y(t, \varepsilon) \| \| z_0 \| + \varepsilon^m M,$$

where

$$M = L \sup \Psi(t, t_1, \varepsilon),$$
$$0 \leqslant t, t_1 \leqslant \frac{L}{\varepsilon}, \quad 0 < \varepsilon \leqslant \varepsilon_0.$$

Now, on the basis of inequality (15.15), we can maintain that we have proved the following theorem.

Theorem 3.2. Let us assume that the exact solution $x(t, \varepsilon)$ of system (12.1) /80 and the approximation of m-th order $x^{(m)}(t, \varepsilon)$ (15.11) to this solution are taken with the same initial conditions. Then if the conditions of Lemma 3.2 are satisfied, we can find a constant M that is independent of ε, such that for all $t \in [0, \frac{L}{\varepsilon}], 0 < \varepsilon \leq \varepsilon_0$ the following estimate is valid:

$$\| x(t, \varepsilon) - x^{(m)}(t, \varepsilon) \| \leqslant \varepsilon^m M. \tag{15.16}$$

Inequality (15.16) points to the asymptotic nature of the formal solution $x^{(m)}(t, \varepsilon)$ and the algorithm for obtaining this solution is described in Section 13.

Section 16. Several Special Cases of Partitioning

The method described in the present chapter makes it possible to partition the initial system of differential equations into isolated subsystems of orders r_k $(k = 1, 2, \ldots, p; \sum_{k=1}^{p} r_k = n)$ in the case when the roots of the characteristic equation for the matrix $A_0(\tau)$ are partitioned into p nonintersecting groups.

The partitioning of the given system is particularly simple in the case when the eigenvalues of the matrix $A_0(\tau)$ are simple over the entire segment $[0, L]$.

Indeed, according to the method presented in this chapter, in this case we can represent the general formal solution of the given system

$$\frac{dx}{dt} = A(\tau, \varepsilon)\, x \tag{16.1}$$

in the form

$$x(t, \varepsilon) = U(\tau, \varepsilon)\, \xi(t, \varepsilon) \equiv \sum_{k=1}^{n} U_k(\tau, \varepsilon)\, \xi_k(t, \varepsilon), \tag{16.2}$$

where the function $\xi_k(t, \varepsilon)$ is the general solution of the first-order equation

$$\frac{d\xi_k}{dt} = \omega_k(\tau, \varepsilon)\, \xi_k, \quad k = 1, 2, \ldots, n. \tag{16.3}$$

while $U_k(\tau, \varepsilon)$ is an n-dimensional vector which is the corresponding column of the transforming matrix $U(\tau, \varepsilon)$.

Thus, in the given case, system (16.1) of the n-th order is partitioned into n independent equations of the first order (16.3) and its general solution may be written in the form /81

$$x(t, \varepsilon) = \sum_{k=1}^{n} c_k U_k(\tau, \varepsilon)\, e^{\int_{t_0}^{t} \omega_k(\tau, \varepsilon) dt},$$

where c_k are arbitrary constants which are determined from the initial conditions.

During the subsequent determination (using the algorithm described in Section 13) of the terms in the asymptotic expansion of function $\omega_k(\tau, \varepsilon)$ and of the vector $U_k(\tau, \varepsilon)$ we shall find that $U_{k0}(\tau)$ is the eigenvector of the matrix $A_0(\tau)$ which corresponds to the eigenvalue $\lambda_k(\tau)$, and $\omega_{k0}(\tau) \equiv \lambda_k(\tau)$.

Furthermore, in the case of simple eigenvalues for the matrix $A_0(\tau)$, the elements of the matrix $Q_s(\tau)$ ($s \geq 1$) (see (13.20)) are determined by means of the following equations:

$$q_{ijs}(\tau) = \frac{f_{ijs}(\tau)}{\lambda_i(\tau) - \lambda_j(\tau)}, \qquad i \neq j; \qquad i, j = 1, 2, \ldots, n,$$

where $f_{ijs}(\tau)$ is the corresponding element of the matrix $F_s(\tau)$ (13.20).

When i=j we can write $q_{iis}(\tau) \equiv 0$, $s \geq 1$ (however $q_{ii0} \equiv 1$!).

If we know the columns of the matrix $Q_s(\tau)$ and utilize equation $U_s(\tau)=V(\tau)Q_s(\tau)$, it is easy to determine the corresponding vectors $U_{ks}(\tau)$.

The case which has been considered shows that by using the algorithm which is described in Section 13 we can, under definite conditions, establish rather simply the numerical value of the transforming matrix $U(\tau,\varepsilon)$, i.e., we can in fact carry out the asymptotic partitioning of a given system of differential equations into several subsystems of a lower order.

However, in the general case when the eigenvalues of the matrix $A_0(\tau)$ have a variable multiplicity for $\tau\in[0,L]$ and the order of the initial system is sufficiently high, the algorithm which has been described meets with significant computation difficulties. In particular, it is not very easy to construct the matrix $U_0(\tau)\equiv V(\tau)$ and also to solve the homogeneous system of algebraic equations of form (13.20) for each $s\geq 1$. Therefore, in such cases the calculations must be carried out by means of high-speed electronic computers which in turn requires the application of a special, more convenient, algorithm. We shall return to this problem in Chapter 5.

Section 17. The Partitioning of a Nonhomogeneous System of Linear Differential Equations

In this section we shall consider the question of the asymptotic partitioning of a system of differential equations of the form /82

$$\frac{dx}{dt} = A(\tau, \varepsilon)x + b(\tau, \varepsilon)e^{i\theta(t,\varepsilon)}, \tag{17.1}$$

where the matrix $A(\tau,\varepsilon)$ is the same as in Sections 13-15, while the n-dimensional vector $b(\tau,\varepsilon)$ is infinitely differentiable with respect to τ on $[0,L]$ and admits of the following representation:

$$b(\tau, \varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s b_s(\tau). \tag{17.2}$$

As in the preceding chapters, we must distinguish between two cases:

a) the case of "resonance," when the values of the function $i\nu(\tau)\equiv i\frac{d\theta}{dt}$ $(i=\sqrt{-1})$ coincide at some $\tau\in[0,L]$ with the values of the roots, for the characteristic equation of the matrix $A_0(\tau)$, which form one isolated group and do not coincide with another group of roots for any τ;

b) the case of "nonresonance," when the function $i\nu(\tau)$ does not assume values coinciding with the eigenvalues of the matrix $A_0(\tau)$ for any $\tau\in[0,L]$.

1. The case of "resonance." Let the values of the function $i\nu(\tau)$ for some $\tau\in[0, L]$ coincide with the values of some eigenvalues from the first group, i.e., with certain $\lambda_i(\tau)$ where $i=1, 2, \ldots, r_1$.

Then the following theorem exists.

Theorem 3.3. When the conditions of Theorem 3.1 are satisfied and when the vector $b(\tau, \varepsilon)$ and the functions $\nu(\tau)$ are infinitely differentiable with respect to τ, then the formal general solution of system (17.1) in the case of "resonance" may be represented in the form

$$x(t, \varepsilon) = U(\tau, \varepsilon)\xi(t, \varepsilon) + P(\tau, \varepsilon)e^{i\theta(t,\varepsilon)}, \tag{17.3}$$

where the matrix $U(\tau, \varepsilon)$ is the same as in Section 13; the n-dimensional vector $\xi(t, \varepsilon)$ is the general solution of the nonhomogeneous equation

$$\frac{d\xi}{dt} = \mathfrak{A}(\tau,\varepsilon)\xi + Z(\tau, \varepsilon)e^{i\theta(t,\varepsilon)}, \tag{17.4}$$

in which the matrix $\mathfrak{A}(\tau, \varepsilon)$ is the same as in Section 13:

$$\mathfrak{A}(\tau, \varepsilon) = \begin{bmatrix} \mathfrak{A}_1(\tau, \varepsilon) & 0 \\ 0 & \mathfrak{A}_2(\tau, \varepsilon) \end{bmatrix}, \tag{17.5}$$

while in the vector $Z(\tau, \varepsilon)$ only the first r_1 components are different from zero

$$Z(\tau, \varepsilon) = \begin{bmatrix} Z_1(\tau, \varepsilon) \\ 0 \end{bmatrix}. \tag{17.6}$$

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The vector $P(\tau, \varepsilon)$ is n-dimensional and, like the vector $Z(\tau, \varepsilon)$, admits expansion in the usual formal manner

$$P(\tau, \varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s P_s(\tau), \qquad Z(\tau, \varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s Z_s(\tau). \tag{17.7}$$

In other words, by using transformation (17.3), which can also be written in the form

$$x(t, \varepsilon) = U_1(\tau, \varepsilon)\xi_1(t, \varepsilon) + U_2(\tau, \varepsilon)\xi_2(t, \varepsilon) + P(\tau, \varepsilon)e^{i\theta(t,\varepsilon)}, \tag{17.8}$$

the nonhomogeneous system of the n-th order (17.1) is asymptotically partitioned into two isolated systems of a lower order. Specifically,

$$\frac{d\xi_1}{dt} = \mathfrak{A}_1(\tau, \varepsilon)\xi_1 + Z_1(\tau, \varepsilon)e^{i\theta(t,\varepsilon)}, \tag{17.9}$$

$$\frac{d\xi_2}{dt} = \mathfrak{A}_2(\tau, \varepsilon)\,\xi_2. \tag{17.10}$$

System (17.9) has the order r_1, and system (17.10) has the order r_2 ($r_1+r_2=n$) in accordance with the number of roots in each of the isolated groups (see Section 12). We shall not present, in detail, the proof of Theorem 3.3, since it is entirely analogous to the proof of Theorem 3.1. In addition to this the equations for determining the terms of the asymptotic expansion of the matrices $U(\tau, \varepsilon)$ and $\mathfrak{A}(\tau, \varepsilon)$ of this section are identical with the corresponding equations of Section 13.

We shall consider only the determination of terms in expansions (17.7).

By following an approach which has already been described in detail, we find that to prove Theorem 3.3 (without concerning ourselves with the construction of matrices $\mathfrak{A}(\tau, \varepsilon)$ and $U(\tau, \varepsilon)$, it is sufficient to select the vectors $P(\tau, \varepsilon)$ and $Z(\tau, \varepsilon)$ such that the following equality is satisfied:

$$\varepsilon \frac{dP}{d\tau} + U(\tau, \varepsilon) Z(\tau, \varepsilon) = [A(\tau, \varepsilon) - i\nu(\tau) E] P(\tau, \varepsilon) + b(\tau, \varepsilon). \tag{17.11}$$

By successively separating the coefficients in front of the powers ε^s ($s=0, 1, 2, \ldots$) in equality (17.11) we obtain recurrence relationships for determining the vectors $Z_s(\tau)$ and $P_s(\tau)$:

$$[A_0(\tau) - i\nu(\tau) E] P_s(\tau) = U_0(\tau) Z_s(\tau) + G_s(\tau), \tag{17.12}$$

where /84

$$G_s(\tau) = \frac{dP_{s-1}}{d\tau} + \sum_{k=0}^{s-1} [U_{s-k}(\tau) Z_k(\tau) - A_{s-k}(\tau) P_k(\tau)] - b_s(\tau), \tag{17.13}$$
$$s = 0, 1, 2, \ldots.$$

Multiplying the left side of (17.12) by $V^{-1}(\tau)$ and introducing the new vector $R_s(\tau)$ according to the equation

$$V^{-1} P_s(\tau) = R_s(\tau), \qquad s = 0, 1, 2, \ldots, \tag{17.14}$$

we obtain

$$[W_0(\tau) - i\nu(\tau) E] R_s(\tau) = Z_s(\tau) + \Phi_s(\tau), \tag{17.15}$$

where

$$\Phi_s(\tau) = V^{-1}(\tau) G_s(\tau), \quad V^{-1}(\tau) U_0(\tau) = Q_0(\tau) \equiv E.$$

According to the structure of matrix W_0 and of the vector $Z_s(\tau)$, system (17.15) is partitioned into two independent systems of lower orders:

$$[W_{k0}(\tau) - i\nu(\tau) E_{r_k}] R_{ks}(\tau) = Z_{ks}(\tau) + \Phi_{ks}(\tau) \quad (k = 1, 2). \tag{17.16}$$

Here we have adopted the following designations:

$$R_s(\tau) = \begin{bmatrix} R_{1s}(\tau) \\ R_{2s}(\tau) \end{bmatrix}, \qquad \Phi_s(\tau) = \begin{bmatrix} \Phi_{1s}(\tau) \\ \Phi_{2s}(\tau) \end{bmatrix}. \tag{17.17}$$

Let k=2. Then (17.16) has the form

$$[W_{20}(\tau) - i\nu(\tau) E_{r_2}] R_{2s}(\tau) = \Phi_{2s}(\tau). \tag{17.18}$$

Since the values of $i\nu(\tau)$ for $\tau \in [0, L]$ do not coincide with the roots of the second group, the matrix $[W_{20}(\tau) - i\nu(\tau) E_{r_2}]$ is nonsingular and equation (17.18) has a unique solution $R_{2s}(\tau)$:

$$R_{2s}(\tau) = [W_{20}(\tau) - i\nu(\tau) E_{r_2}]^{-1} \Phi_{2s}(\tau), \quad s = 0, 1, 2, \ldots. \tag{17.19}$$

When k=1, system (17.16) assumes the form

$$[W_{10}(\tau) - i\nu(\tau) E_{r_1}] R_{1s}(\tau) = Z_{1s}(\tau) + \Phi_{1s}(\tau), \tag{17.20}$$

from which it is necessary to determine the unknown vectors $R_{1s}(\tau)$ and $Z_{1s}(\tau)$.

In view of our propositions, matrix $[W_{10}(\tau) - i\nu(\tau) E_{r_1}]$ becomes singular at some points on the segment $[0, L]$:

$$\det [W_{10}(\tau) - i\nu(\tau) E] = 0. \tag{17.21}$$

Therefore we must again remember the necessary and sufficient condition for the solution of nonhomogeneous algebraic systems whose determinants are equal to zero (see Section 7). Although the determinant (17.21) becomes equal to zero only at certain points $\tau \in [0, L]$, for the present we shall select an unknown vector $Z_{1s}(\tau)$ so that the above condition of solvability is satisfied over the entire segment $[0, L]$. Specifically, we do this so that the right side of equation (17.20) $f_s(\tau) = Z_{1s}(\tau) + \Phi_{1s}(\tau)$ be orthogonal to all solutions of the algebraic system

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$$[W_{10}^*(\tau) + i\nu(\tau) E_{r_1}] y = 0.$$

The formulated condition will be satisfied beforehand if we select $Z_{1s}(\tau)$ in such a way that

$$Z_{1s}(\tau) + \Phi_{1s}(\tau) = 0,$$

i.e.,

$$Z_{1s}(\tau) = -\Phi_{1s}(\tau), \quad s = 0, 1, 2, \ldots.$$

In view of our selection of vector $Z_{1s}(\tau)$ we must limit ourselves to the trivial solution for the vector $R_{1s}(\tau)$ on the segment [0,L]. Specifically, we must let

$$R_{1s}(\tau) \equiv 0, \quad s = 0,1, 2\ldots.$$

Thus, during the successive solution of system (17.16) we can determine, by taking into account (17.14), any term in the expansions for the unknown vectors $Z(\tau, \varepsilon)$ and $P(\tau, \varepsilon)$. This proves, in conjunction with Theorem 3.1, that Theorem 3.3 is valid.

2. The partitioning of system (17.1) becomes even simpler in the case of "nonresonance." Here we can formulate and prove a theorem which is analogous to Theorem 3.3, i.e., a theorem which shows that in the case of "nonresonance" the general formal solution of system (17.1) may be represented in the form

$$x(t, \varepsilon) = U(\tau, \varepsilon)\xi(t, \varepsilon) + H(\tau, \varepsilon)e^{i\theta(t, \varepsilon)}, \qquad (17.22)$$
$$\frac{d\xi}{dt} = \mathfrak{A}(\tau, \varepsilon)\xi,$$

where the matrices $U(\tau, \varepsilon)$ and $\mathfrak{A}(\tau, \varepsilon)$ are the same as in Theorems 3.1 and 3.3, while the n-dimensional vector $H(\tau, \varepsilon)$ admits a formal expansion

$$H(\tau, \varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s H_s(\tau). \qquad (17.23)$$

The recurrence equations for determining the terms of series (17.23) may be obtained if we seek the particular solution of the nonhomogeneous system (17.1) in the form

$$X(\tau, \varepsilon) = H(\tau, \varepsilon)e^{i\theta}. \qquad (17.24)$$

Indeed, by substituting (17.24) into system (17.1) we find an equation which must be satisfied by the unknown vector $H(\tau, \varepsilon)$: /86

$$[A(\tau, \varepsilon) - i\nu(\tau)E]H(\tau, \varepsilon) = \varepsilon\frac{dH}{d\tau} - b(\tau, \varepsilon). \qquad (17.25)$$

From this, according to the formal expansions, we seek a place for all terms of equation (17.25) and obtain the following recurrence equations for determining the terms of series (17.23):

$$[A_0(\tau) - i\nu(\tau)E]H_s(\tau) = \Psi_s(\tau), \quad s = 0, 1, 2, \ldots, \qquad (17.26)$$

where

$$\Psi_s(\tau)=\frac{dH_{s-1}}{d\tau}-b_s(\tau)-\sum_{k=0}^{s-1}A_{s-k}(\tau)H_k(\tau). \qquad (17.26')$$

For the case of "nonresonance," system (17.26) for each $s\geq 0$ determines a unique solution

$$H_s(\tau)=[A_0(\tau)-iv(\tau)E]^{-1}\Psi_s(\tau). \qquad (17.27)$$

Thus, in the case considered, where there is no "resonance" the derivation of formal solution (17.22) for the nonhomogeneous system (17.1) is reduced to the solution of two independent differential systems of lower orders:

$$\frac{d\xi_1}{dt}=\mathfrak{A}_1(\tau,\varepsilon)\xi_1,\qquad \frac{d\xi_2}{dt}=\mathfrak{A}_2(\tau,\varepsilon)\xi_2,\qquad \xi(t,\varepsilon)=\begin{bmatrix}\xi_1(t,\varepsilon)\\ \xi_2(t,\varepsilon)\end{bmatrix}$$

and the determination of vector $H(\tau,\varepsilon)$ by means of recurrence equations (17.26).

Following the method of Section 15 we can show that the formal solutions of the nonhomogeneous system (17.1) which we have constructed are asymptotic in both cases ("resonance" and "nonresonance"). In the case of "resonance" the estimate has the form

$$\|x(t,\varepsilon)-x^{(m)}(t,\varepsilon)\|\leq C\varepsilon^{m-1},$$

while for the case of "nonresonance"

$$\|x(t,\varepsilon)-x^{(m)}(t,\varepsilon)\|\leq C_1\varepsilon^{m}.$$

The last improved estimate is valid for both cases if $b_0(\tau)\equiv 0$.

Note 1. It is easy to see that all the constructions made in this section are valid in the case when the roots of the characteristic equation for the matrix $A_0(\tau)$ are partitioned into p isolated groups. The modification and the structure of matrix $\mathfrak{A}(\tau,\varepsilon)$ and of the vector $Z(\tau,\varepsilon)$ (in the case of "resonance") are obvious.

Note 2. The asymptotic partitioning of a system of higher order into several isolated subsystems which we have constructed in this chapter do not always make it possible to write out the asymptotic solution of the initial system of differential equations in a direct manner. This is explained by the fact that the partitioned systems of equations, although they are of a lower order than the given system, cannot always be integrated in closed form. /87

Only when we have simple eigenvalues of the matrix $A_0(\tau)$ (see Section 16) is it possible to obtain the solution by quadratures. When we have multiple roots of the characteristic equation, the construction of asymptotic solutions becomes possible only in individual cases and generally speaking is substantially complicated.

These special cases will be considered in the fourth chapter.

Section 18. The Asymptotic Partitioning of a System of Ordinary Linear Differential Equations in the Case When the Roots of the Characteristic Equation are Multiple

In the preceding sections of Chapter 3 we have shown that when definite conditions are satisfied it is possible to have an asymptotic partitioning of a system of linear differential equations of a high order into several independent subsystems of a lower order.

In this case, in speaking of asymptotic partitioning we had in mind the asymptotic nature of the convergence of the approximate solution $x^{(m)}(t,\varepsilon)$ to an exact solution of the initial system $x(t,\varepsilon)$.

However, there is another approach to the problem of determining the possibility of an asymptotic partitioning of a system of linear differential equations, specifically: in contrast to what has been said, it is possible to prove the asymptotic convergence of the transforming matrix into some "exact" matrix which partitions the given system.

Such an approach, for example, is contained in references 126, 29, and 30. In the present section, which is based on the works of A. G. Ilyukhin, we present the salient feature of this second approach.

1. We again consider the system of linear differential equations (12.1):

$$\frac{dx}{dt} = A(\tau, \varepsilon)x. \tag{18.1}$$

We shall assume that the matrix $A(\tau,\varepsilon)$ satisfies condition 1 of Section 12, while condition 2 is replaced by the following.

The roots $\lambda_1(\tau),\lambda_2(\tau), \ldots, \lambda_n(\tau)$ of the characteristic equation (12.3) may be broken down into p classes $p\leq n$, each of which contains n_j ($j=1, 2, \ldots, p$; $\sum_{j=1}^{p} n_j = n$) multiple roots, and the multiplicity of the roots is retained over /88 the entire interval $0\leq\tau\leq L$. In addition to this it is assumed that the matrix $A_0(\tau)$ retains its canonical structure over the entire interval $[0,L]$, i.e., for

any $\tau\in[0,L]$ the matrix $A_0(\tau)$ is reduced to a normal form with the same number of Jordan cells along the main diagonal.

Then we can construct a nonsingular matrix $V(\tau)$ of order n, which will transform the matrix $A_0(\tau)$ into the matrix $W_0(\tau)$ of the canonical Jordan form

$$W_0(\tau)=V^{-1}(\tau)A_0(\tau)V(\tau), \tag{18.2}$$

where

$$W_0(\tau)=\begin{bmatrix} W_{10}(\tau) & 0 & \dots & 0 \\ 0 & W_{20}(\tau) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & W_{p0}(\tau) \end{bmatrix}, \tag{18.3}$$

while $W_{j0}(\tau)(j=1, 2, \dots, p)$ are cells of the form

$$W_{j0}(\tau)=\begin{bmatrix} \lambda_j(\tau) & \alpha_{j1} & \dots & 0 & 0 \\ 0 & \lambda_j(\tau) & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_j(\tau) & \alpha_{j,n_j-1} \\ 0 & 0 & \dots & 0 & \lambda_j(\tau) \end{bmatrix}, \tag{18.4}$$

where the numbers $\alpha_{j1}, \dots, \alpha_{j,nj-1}$ may be either zero or unity for all $\tau\in[0,L]$ in accordance with the proposition adopted above. Thus, in $W_{j0}(\tau)(j=1, 2, \dots, p)$ all the Jordan cells corresponding to the root $\lambda_j(\tau)$ of multiplicity $n_j(j=1, 2, \dots, p)$ are combined.

In this case the matrix $V(\tau)$ turns out to be differentiable the same number of times as $A_0(\tau)$. Indeed, in Section 14 it was shown that if the spectrum of matrix $A_0(\tau)$ breaks down into p isolated parts when $\tau\in[0,L]$, we can specify a certain matrix $T(\tau)$ which reduces $A_0(\tau)$ to a cell-diagonal form $\Lambda(\tau)=[\Lambda_1(\tau), \Lambda_2(\tau), \dots, \Lambda_p(\tau)]$ where $\Lambda_j(\tau)(j=1, 2, \dots, n)$ are diagonal cells whose orders are determined by the dimensions of the invariant subsystems which correspond to the isolated parts of the spectrum for matrix $A_0(\tau)$. It was also proven in Section 14 that $T(\tau)$ and $\Lambda_j(\tau)(j=1, 2, \dots, p)$ have as many derivatives to τ as does $A_0(\tau)$.

By using analogous methods we shall now show that it is possible to construct a matrix $\tilde{V}(\tau)$ which under our propositions concerning $A_0(\tau)$ reduces the cell diagonal matrix $\Lambda(\tau)$ to the form (18.3). In this case $\tilde{V}(\tau)$ will be differentiable the same number of times as the matrices $\Lambda_j(\tau)$ (j=1, 2, ..., p), i.e., /89 in the final analysis, the same number of times as the matrix $A(\tau)$.

Indeed, to be certain, let us consider the invariant subspace E_1 which corresponds to the first diagonal cell $\Lambda_1(\tau)$ of the matrix $\Lambda(\tau)$. The subspace E_1 of dimensionality n_1 in this case will be a root subspace (see reference 49) corresponding to the root $\lambda_1(\tau)$. Then, by following the well-known methods of linear algebra (see reference 68) we can determine the base in such a way that the matrix $\Lambda(\tau)$ will be reduced to the canonical Jordan form (18.4). To do this we find the smallest $l<n_1$ for which the following matrix equality exists:

$$(\Lambda_1(\tau) - \lambda_1(\tau) E_{n_1})^l = 0. \tag{18.5}$$

To reduce the calculation steps we let

$$\Lambda_1(\tau) - \lambda_1(\tau) E_{n_1} = B(\tau). \tag{18.6}$$

We find all of the nonzero solutions of the homogeneous system of linear algebraic equations

$$B^{l-1}(\tau) x = 0. \tag{18.7}$$

Since the matrix $A_0(\tau)$ has a constant canonical structure for all $\tau\in[0,L]$, the resulting vector solutions of system (18.7) will have the same properties of differentiability as the matrix $B(\tau)$, i.e., in the final analysis they will have the differentiability properties of $\Lambda_1(\tau)$ and of the eigenvalue $\lambda_1(\tau)$.

Indeed, since the multiplicity of $\lambda_1(\tau)$ is the same for any value of τ, the matrix $\Lambda_1(\tau)$, as we have shown above, can be differentiated the same number of times as the matrix $A_0(\tau)$. It can also be shown that the roots $\lambda_j(\tau)$ (j=1, 2, ..., p) have the same number of derivatives as the elements of the matrix $A_0(\tau)$. Then it follows from equality (18.6) that $B(\tau)$ has the same properties of differentiability as the matrix $A_0(\tau)$.

In order that the vector solution of the homogeneous system (18.7) be differentiable the same number of times as the matrix $A_0(\tau)$, it is necessary

additionally, that the matrix $B^{l-1}(\tau)$ have the same rank for all $\tau\in[0,L]$. In turn, the rank of matrix $B^{l-1}(\tau)$ is retained when τ is varied if the canonical form of the matrix $\Lambda_1(\tau)$ is retained, i.e., if the multiplicity of elementary divisors corresponding to the root $\lambda_1(\tau)$ is retained.

Let us illustrate the above by means of a simple example. Let the matrix $\Lambda_1(\tau)$ have the following canonical form for all $\tau\in[0,L]$:

$$\begin{bmatrix} \lambda_1(\tau) & 1 & 0 & 0 \\ 0 & \lambda_1(\tau) & 1 & 0 \\ 0 & 0 & \lambda_1(\tau) & 1 \\ 0 & 0 & 0 & \lambda_1(\tau) \end{bmatrix}. \tag{18.8}$$

Then according to (18.6) the matrix $B(\tau)$ will be equivalent to the following matrix anywhere in the interval $[0,L]$: /90

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{18.9}$$

In this case, for any $\tau\in[0,L]$ the rank of matrix $B^3(\tau)$ will be equal to unity.

If on the other hand, for example, the matrix $\Lambda_1(\tau_1)$ has a canonical form (18.8) for one value of $\tau_1\in[0,L]$, and for another value of $\tau_2\in[0,L]$ the canonical form $\Lambda_1(\tau_2)$ has the form

$$\begin{bmatrix} \lambda_1(\tau_2) & 1 & 0 & 0 \\ 0 & \lambda_1(\tau_2) & 0 & 0 \\ 0 & 0 & \lambda_1(\tau_2) & 1 \\ 0 & 0 & 0 & \lambda_1(\tau_2) \end{bmatrix}, \tag{18.10}$$

the matrix $B^3(\tau_2)$ will be of a zero rank. Consequently, in this second case not all vector solutions for the homogeneous system of equations

$$B^3(\tau)x = 0$$

will be continuous. At least one of these solutions will have a discontinuity in the interval $\tau_1<\tau<\tau_2$.

Thus in order that the vector solution of system (18.7) be infinitely differentiable, two conditions become significant: the infinite differentiability of matrix $A_0(\tau)$ and the constancy of the canonical structure for the matrix $A_0(\tau)$ over the entire interval $[0,L]$. Therefore we assume from the very beginning that these conditions are satisfied.

In constructing the transformation matrix $V_1(\tau)$ which reduces the matrix $\Lambda_1(\tau)$ to the canonical Jordan form, it becomes necessary to find not the solutions of the homogeneous system (18.7) but rather the vectors which do not belong to the subspace of solutions for this system. Thus, since the vector solutions in question are infinitely differentiable, we can select the base in the orthogonal supplement to the subspace of solutions for system (18.7) such that it consists of any number of smooth vector functions. If we apply to the vectors of this space a linear transformation given by the matrix (18.6), we can construct the columns of the matrix $V_1(\tau)$ by means of well-known rules (ref. 68). We shall describe this process briefly. The details of the method are presented in the "Course on Higher Mathematics" (ref. 68).

Thus we shall find the columns of the matrix $V_1(\tau)$ which reduces the matrix $\Lambda_1(\tau)$ to the canonical Jordan form. The number of linearly independent vectors which do not satisfy system (18.7) is obviously equal to the rank of the matrix $B^{l-1}(\tau)$. We designate this number by r_1. Then r_1 linearly independent vectors in a subspace which is orthogonal to the subspace containing the solutions of system (18.7) may be taken as the columns of matrix $V_1(\tau)$. To each of these vectors we apply the transformation given by the matrix $B(\tau)$. Then we again obtain r_1 vectors, some of which may be identically equal to zero while the others will differ from zero. To the latter we again apply the matrix $B(\tau)$, etc., until all the vectors, after a series of matrix $B(\tau)$ applications, become identically equal to zero. /91

Thus each of the r_1 initially selected linearly independent vectors $x^{(k)}(\tau)(k=1, 2, \ldots, r_1)$ together with the vectors $B(\tau)x^{(k)}(\tau), B^2(\tau)x^{(k)}(\tau)$ $\ldots(k=1, 2, \ldots, r_1)$ forms a series of orthogonal unit vectors in the orthogonal supplement to the subspace containing the solutions for system (18.7). A series of linearly independent vectors constructed in this manner (there will be a total of n_1 vectors where n_1 is the order of matrix $\Lambda_1(\tau)$) will form n_1 columns of the unknown matrix $V_1(\tau)$.

In the same way we construct nonsingular square matrices $V_2(\tau), V_3(\tau), \ldots, V_p(\tau)$ of dimensions $n_2, n_3, \ldots, n_p$, respectively, which leads to a normal Jordan form of matrices $\Lambda_2(\tau), \Lambda_3(\tau), \ldots, \Lambda_p(\tau)$.

Furthermore let us construct a square quasi-diagonal matrix $\tilde{V}(\tau)$ of order $n=n_1+n_2+\ldots+n_p$, whose diagonal cells are the matrices $V_1(\tau), V_2(\tau), \ldots, V_p(\tau)$. Then the resulting matrix $\tilde{V}(\tau)=[V_1(\tau), V_2(\tau), \ldots, V_p(\tau)]$ reduces the quasi-diagonal matrix $\Lambda(\tau)=[\Lambda_1(\tau), \Lambda_2(\tau), \ldots, \Lambda_p(\tau)]$ to a normal Jordan form, i.e., we have the following equality

$$\tilde{V}^{-1}(\tau)\,\Lambda(\tau)\tilde{V}(\tau)=W_0(\tau),$$

where $W_0(\tau)$ has the form (18.3).

The process which we have described for the construction of $V_1(\tau), V_2(\tau), \ldots, V_p(\tau)$ shows that the matrix $\tilde{V}(\tau)$ and consequently the matrix $V(\tau)=T(\tau)\tilde{V}(\tau)$, which reduces the matrix $A_0(\tau)$ to the Jordan form (18.3), can be differentiated the same number of times with respect to τ as the matrix $A_0(\tau)$, which was to be proven.

2. As in Section 13, we shall seek the solution of system (18.1) in the form

$$x=U(\tau,\varepsilon)\zeta, \tag{18.11}$$

where $\zeta(t,\varepsilon)$ is an n-dimensional vector which is determined from the system of equations

$$\frac{d\zeta}{dt}=\mathfrak{A}(\tau,\varepsilon)\zeta. \tag{18.12}$$

Here $U(\tau,\varepsilon)$ and $\mathfrak{A}(\tau,\varepsilon)$ are matrices of the n-th order, infinitely differentiable with respect to τ and which can be represented in the form of formal series /92

$$U(\tau,\varepsilon)=\sum_{s=0}^{\infty}\varepsilon^s U_s(\tau), \tag{18.13}$$

$$\mathfrak{A}(\tau,\varepsilon)=\sum_{s=0}^{\infty}\varepsilon^s \mathfrak{A}_s(\tau). \tag{18.14}$$

In accordance with the structure of $W_0(\tau)$ we seek a matrix $\mathfrak{A}(\tau, \varepsilon)$ in the form

$$\mathfrak{A}(\tau, \varepsilon) = \begin{bmatrix} \mathfrak{A}_1(\tau, \varepsilon) & 0 & \dots & 0 \\ 0 & \mathfrak{A}_2(\tau, \varepsilon) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \mathfrak{A}_p(\tau, \varepsilon) \end{bmatrix}, \tag{18.15}$$

where $\mathfrak{A}_j(\tau, \varepsilon)$ $(j=1, 2, \dots, p)$ are square matrices of the same order as the matrix $W_{j0}(\tau, \varepsilon)$ respectively, and where an expansion of form (18.14) exists for $\mathfrak{A}_j(\tau, \varepsilon)$ $(j=1, 2, \dots, p)$.

As in Section 13, we obtain the following matrix equation for the determination of the matrices $U(\tau, \varepsilon), \mathfrak{A}(\tau, \varepsilon)$:

$$\varepsilon \frac{dU(\tau, \varepsilon)}{d\tau} + U(\tau, \varepsilon)\mathfrak{A}(\tau, \varepsilon) - A(\tau, \varepsilon)U(\tau, \varepsilon) = 0. \tag{18.16}$$

We introduce the matrices $Q(\tau, \varepsilon)$ and $W(\tau, \varepsilon)$ which are determined from the equalities

$$Q(\tau, \varepsilon) = V^{-1}(\tau)U(\tau, \varepsilon), \tag{18.17}$$

$$W(\tau, \varepsilon) = V^{-1}(\tau)A(\tau, \varepsilon)V(\tau). \tag{18.18}$$

Since according to the proposition, the matrices $A(\tau, \varepsilon)$ and $U(\tau, \varepsilon)$ can be represented in power series of ε, it is obvious that the matrices $Q(\tau, \varepsilon)$ and $W(\tau, \varepsilon)$ have the same properties:

$$Q(\tau, \varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s Q_s(\tau), \tag{18.19}$$

$$W(\tau, \varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s W_s(\tau). \tag{18.20}$$

Multiplying (18.16) by $V^{-1}(\tau)$ on the left side we obtain a matrix equation

$$\varepsilon V^{-1}(\tau)\frac{dV}{d\tau}Q(\tau, \varepsilon) + \varepsilon \frac{dQ(\tau, \varepsilon)}{d\tau} + Q(\tau, \varepsilon)\mathfrak{A}(\tau, \varepsilon) - W(\tau, \varepsilon)Q(\tau, \varepsilon) = 0, \tag{18.21}$$

from which we can determine the expansion terms of the unknown matrices $Q(\tau, \varepsilon)$ and $\mathfrak{A}(\tau, \varepsilon)$ by using methods presented in Section 13. /93

3. In the subsequent presentation we shall require equations (13.12)-(13.14) which can also be obtained from (18.21).

We partition each of the matrices $Q_s(\tau)$ and $F_s(\tau)$ contained in equation (13.14) into cells in such a way that the principal diagonals contain the square cells of order $n_1, n_2, \ldots, n_p$ $\left(\sum_{j=1}^{p} n_j = n\right)$, which are equal to the orders of the corresponding diagonal cells for the matrices $\mathfrak{A}_s(\tau)$ $(s=1, 2, \ldots)$.

As a result we shall obtain cell matrices which have a more general form compared with (13.16) (in our case the matrices will consist of p^2 cells). We note that the matrices $\mathfrak{A}_s(\tau)$ $(s=1, 2, \ldots)$ and also $W_0(\tau)$, as we can see from (18.15) and (18.3), already have the required cell structure.

After this we can repeat all the arguments of Section 13 which pertain to the solution of the matrix equation (13.14), i.e., to the determination of matrices $Q_s(\tau)$ and $\mathfrak{A}_s(\tau)$ $(s=1, 2, \ldots)$.

However, it will be more convenient to deviate from (13.18) and let

$$Q_{iis}(\tau) \equiv 0 \quad (i = 1, 2, \ldots, p;\ s = 1, 2, \ldots). \tag{18.22}$$

This is entirely permissible because the determination of the diagonal cells of matrices $Q_s(\tau)$ $(s=1, 2, \ldots)$ is associated with a certain arbitrariness, as we can see from system (13.17).

Then the remaining cells $Q_{ijs}(\tau)$ $(i, j=1, 2, \ldots, p, i \neq j;\ s=1, 2, \ldots)$, and also $\mathfrak{A}_{(s)}(\tau)$ $(s=1, 2, \ldots)$ are uniquely determined from the matrix equations (13.14).

Thus, according to (18.19) we shall obtain a transformation matrix $Q(\tau, \varepsilon)$ in the form of a formal matrix series whose asymptotic convergence will be shown below.

4. In the future we shall require several preliminary constructions.

Taking into account (18.19), (18.20), (13.12) and (13.13), we represent each of the matrices $W(\tau, \varepsilon)$, $\mathfrak{A}(\tau, \varepsilon)$ and $Q(\tau, \varepsilon)$ in the form of two terms

$$W(\tau, \varepsilon) = W_0(\tau) + \widetilde{W}(\tau, \varepsilon), \tag{18.23}$$

$$\mathfrak{A}(\tau, \varepsilon) = W_0(\tau) + \widetilde{\mathfrak{A}}(\tau, \varepsilon), \tag{18.24}$$

$$Q(\tau, \varepsilon) = E + \widetilde{Q}(\tau, \varepsilon), \tag{18.25}$$

where

$$\widetilde{W}(\tau, \varepsilon) = \sum_{s=1}^{\infty} \varepsilon^s W_s(\tau), \tag{18.26}$$

$$\widetilde{\mathfrak{A}}(\tau, \varepsilon) = \sum_{s=1}^{\infty} \varepsilon^s \mathfrak{A}_s(\tau), \tag{18.27}$$

$$\widetilde{Q}(\tau, \varepsilon) = \sum_{s=1}^{\infty} \varepsilon^s Q_s(\tau). \tag{18.28}$$

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We write $\widetilde{W}(\tau, \varepsilon)$ in the form of a cell matrix containing the matrices $\widetilde{W}_{ii}(\tau, \varepsilon)$ of order n_i $(i=1, 2, \ldots, p)$ along the principal diagonal:

$$\widetilde{W}(\tau, \varepsilon) = \begin{bmatrix} \widetilde{W}_{11}(\tau, \varepsilon) & \widetilde{W}_{12}(\tau, \varepsilon) & \ldots & \widetilde{W}_{1p}(\tau, \varepsilon) \\ \widetilde{W}_{21}(\tau, \varepsilon) & \widetilde{W}_{22}(\tau, \varepsilon) & \ldots & \widetilde{W}_{2p}(\tau, \varepsilon) \\ \ldots & \ldots & \ldots & \ldots \\ \widetilde{W}_{p1}(\tau, \varepsilon) & \widetilde{W}_{p2}(\tau, \varepsilon) & \ldots & \widetilde{W}_{pp}(\tau, \varepsilon) \end{bmatrix}. \tag{18.29}$$

According to condition (18.22) adopted above, which must be satisfied by the matrices $Q_s(\tau)$ $(s=1, 2, \ldots)$, the matrix $\widetilde{Q}(\tau, \varepsilon)$ may be broken down into cells in the following manner:

$$\widetilde{Q}(\tau, \varepsilon) = \begin{bmatrix} 0 & \widetilde{Q}_{12}(\tau, \varepsilon) & \ldots & \widetilde{Q}_{1p}(\tau, \varepsilon) \\ \widetilde{Q}_{21}(\tau, \varepsilon) & 0 & \ldots & \widetilde{Q}_{2p}(\tau, \varepsilon) \\ \ldots & \ldots & \ldots & \ldots \\ \widetilde{Q}_{p1}(\tau, \varepsilon) & \widetilde{Q}_{p2}(\tau, \varepsilon) & \ldots & 0 \end{bmatrix}, \tag{18.30}$$

where the zeros on the principal diagonal signify zero matrices of order n_j $(j=1, 2, \ldots, p)$.

The matrix $V^{-1}(\tau) \frac{dV(\tau)}{d\tau}$ can also be represented in the form of a cell matrix by analogy with (18.29) and (18.30). We shall designate its cells by

$$\left(V^{-1}(\tau)\frac{dV(\tau)}{d\tau}\right)_{jk} \quad (j, k = 1, 2, \ldots, p). \tag{18.31}$$

Taking into account equalities (18.23)-(18.25), we write matrix equation (18.21) in a slightly different form

$$\varepsilon V^{-1}(\tau)\frac{dV(\tau)}{d\tau}[E + \widetilde{Q}(\tau, \varepsilon)] + \varepsilon\frac{d\widetilde{Q}(\tau,\varepsilon)}{d\tau} + \tag{18.32}$$
$$+ [E + \widetilde{Q}(\tau, \varepsilon)][W_0(\tau) + \widetilde{\mathfrak{A}}(\tau, \varepsilon)] - [W_0(\tau) + \widetilde{W}(\tau, \varepsilon)][E+\widetilde{Q}(\tau, \varepsilon)]=0.$$

All the matrices contained in equation (18.32) are broken down in the same manner in p^2 cells. Therefore in (18.32) we can multiply the cell matrices. Writing out the results of such a multiplication for diagonal and nondiagonal cells, we obtain the following relationships

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$$\widetilde{\mathfrak{A}}_j(\tau, \varepsilon) = \widetilde{W}_{jj}(\tau, \varepsilon) + \sum_{k=1}^{p}\widetilde{W}_{jk}(\tau, \varepsilon)\widetilde{Q}_{kj}(\tau, \varepsilon) -$$
$$-\varepsilon\left(V^{-1}(\tau)\frac{dV(\tau)}{d\tau}\right)_{jj} - \varepsilon\sum_{k=1}^{p}\left(V^{-1}(\tau)\frac{dV(\tau)}{d\tau}\right)_{jk}\widetilde{Q}_{kj}(\tau, \varepsilon), \; j=1, 2, \ldots, p, \tag{18.33}$$

and

$$\varepsilon\frac{d\widetilde{Q}_{jk}(\tau, \varepsilon)}{d\tau} = W_{j0}(\tau)\widetilde{Q}_{jk}(\tau, \varepsilon) - \widetilde{Q}_{jk}(\tau, \varepsilon)W_{k0}(\tau) +$$
$$+\widetilde{W}_{jk}(\tau, \varepsilon) - \widetilde{Q}_{jk}(\tau,\varepsilon)\widetilde{\mathfrak{A}}_k(\tau,\varepsilon) + \sum_{l=1}^{p}W_{jl}(\tau, \varepsilon)\widetilde{Q}_{lk}(\tau, \varepsilon) - \tag{18.34}$$
$$-\varepsilon\left(V^{-1}(\tau)\frac{dV(\tau)}{d\tau}\right)_{jk} - \varepsilon\sum_{l=1}^{p}\left(V^{-1}(\tau)\frac{dV(\tau)}{d\tau}\right)_{jk}\widetilde{Q}_{lk}(\tau, \varepsilon)$$

when $j\neq k$ $(j,k=1, 2, \ldots, p)$.

By substituting the values of $\widetilde{\mathfrak{A}}_k(\tau,\varepsilon)$ from relationships (18.33) into (18.34) we obtain

$$\varepsilon\frac{d\widetilde{Q}_{jk}(\tau, \varepsilon)}{d\tau} = W_{j0}(\tau)\widetilde{Q}_{jk}(\tau, \varepsilon) - \widetilde{Q}_{jk}(\tau, \varepsilon)W_{k0}(\tau) +$$
$$+\sum_{l=1}^{p}W_{jl}(\tau,\varepsilon)\widetilde{Q}_{lk}(\tau, \varepsilon) - \varepsilon\sum_{l=1}^{p}\left(V^{-1}(\tau)\frac{dV(\tau)}{d\tau}\right)_{jl}\widetilde{Q}_{lk}(\tau, \varepsilon) +$$

$$+\widetilde{W}_{jk}(\tau,\varepsilon)-\varepsilon\left(V^{-1}(\tau)\frac{dV(\tau)}{d\tau}\right)_{jk}-\widetilde{Q}_{jk}(\tau,\varepsilon)\widetilde{W}_{kk}(\tau,\varepsilon)-$$

$$-\widetilde{Q}_{jk}(\tau,\varepsilon)\sum_{l=1}^{p}\widetilde{W}_{kl}(\tau,\varepsilon)\widetilde{Q}_{lk}(\tau,\varepsilon)+$$

$$+\varepsilon\widetilde{Q}_{jk}(\tau,\varepsilon)\sum_{l=1}^{p}\left(V^{-1}(\tau)\frac{dV(\tau)}{d\tau}\right)_{kl}\widetilde{Q}_{lk}(\tau,\varepsilon)+ \qquad (18.35)$$

$$+\varepsilon\widetilde{Q}_{jk}(\tau,\varepsilon)\left(V^{-1}(\tau)\frac{dV(\tau)}{d\tau}\right)_{kk}\quad(j\neq k;\ j,k=1,2,\ldots,p).$$

Thus we have obtained a system (18.35) of ordinary nonlinear differential equations which must be satisfied by the elements of matrices $\widetilde{Q}_{jk}(\tau,\varepsilon)$ ($j\neq k$; $j,k=1,2,\ldots,p$). If we write system (18.35) in an expanded form it is easy to see that the order of this system will be equal to $n^2-\sum_{j=1}^{p}n_j^2$; in addition, the right sides of the system will be multinomials of the second degree with respect to the unknown functions (with respect to the elements of the matrix $\widetilde{Q}(\tau,\varepsilon)$) with coefficients which depend on τ and ε. Since these coefficients are constructed from the elements of matrices $\widetilde{W}(\tau,\varepsilon)$ and $V^{-1}(\tau)\frac{dV(\tau)}{d\tau}$ (see (18.35)), they can be expanded in power series of ε, as the elements of matrix $\widetilde{W}(\tau,\varepsilon)$. /96

5. Obviously system (18.35) is satisfied by the formal solution

$$\widetilde{Q}_{jk}(\tau,\varepsilon)=\sum_{s=1}^{\infty}\varepsilon^s\widetilde{Q}_{jks}(\tau)\quad(j\neq k;\ j,k=1,2,\ldots p), \qquad (18.36)$$

which is obtained by means of the recurrence process (13.14) described in Section 13, because when we derived systems (18.35) we proceeded from equations (13.14). Now we shall show that this formal solution is an asymptotic expansion of a certain exact solution of the system of differential equations (18.35). In this way we shall show the asymptotic convergence of the formal series (18.36) to some "exact" transformation matrix $Q(\tau,\varepsilon)$.

Before formulating the basic theorem of this section which pertains to the asymptotic convergence of series (18.36), we consider system (18.35) in more detail. For the sake of brevity, in the future we shall designate the unknown elements of matrices ($j\neq k$; $j,k=1,2,\ldots,p$) by $u_1,u_2,\ldots,u_m$. Above we have noted that the order of system (18.35) is equal to

$$m=n^2-\sum_{j=1}^{p}n_j^2. \qquad (18.37)$$

The first two matrix terms in the right side of (18.35) in a certain sense played a main role because they are linear with respect to $u_1, u_2, \ldots, u_m$ and /97

the coefficients of the unknown functions in these terms do not depend on ε. The remaining terms (both linear and nonlinear) in the right sides of equations (18.35) are infinitesimals of order ε. Indeed, each of these terms, according to (18.26) and (18.35), has the form

$$\varepsilon F_k(\tau, \varepsilon) u_k \text{ or } \varepsilon G_{jk}(\tau, \varepsilon) u_j u_k \qquad (j, k = 1, \ldots, p),$$

where $F_k(\tau, \varepsilon)$ and $G_{jk}(\tau, \varepsilon)$ are certain functions of τ and ε which are infinitely differentiable with respect to τ on the segment $[0, L]$ and which are majorized by constants which are independent of ε.

We designate the principal linear part of system (18.35) by expressions

$$W_{j0}(\tau)\widetilde{Q}_{jk}(\tau, \varepsilon) - \widetilde{Q}_{jk}(\tau, \varepsilon) W_{k0}(\tau) \qquad (18.38)$$
$$(j \neq k;\ j, k = 1, 2, \ldots, p).$$

For simplicity we shall illustrate the transformation of system (18.35) by means of an example. In this case all of the arguments pertaining to the example can be directly transposed to the general case.

6. Thus let us assume that our initial system of differential equation (18.1) has an order n=5, and its corresponding matrix (18.3) consists of two Jordan cells:

$$W_{10} = \begin{bmatrix} \lambda_1(\tau) & 1 & 0 \\ 0 & \lambda_1(\tau) & 1 \\ 0 & 0 & \lambda_1(\tau) \end{bmatrix}, \qquad W_{20}(\tau) = \begin{bmatrix} \lambda_2(\tau) & 1 \\ 0 & \lambda_2(\tau) \end{bmatrix}. \qquad (18.39)$$

For this case the unknown elements ($\widetilde{Q}_{12}$ and $\widetilde{Q}_{21}$) of the matrix $\widetilde{Q}_1(\tau, \varepsilon)$ in system (18.35) will have the form

$$\widetilde{Q}_{12}(\tau, \varepsilon) = \begin{bmatrix} u_1 & u_4 \\ u_2 & u_5 \\ u_3 & u_6 \end{bmatrix}, \qquad \widetilde{Q}_{21}(\tau, \varepsilon) = \begin{bmatrix} u_7 & u_8 & u_9 \\ u_{10} & u_{11} & u_{12} \end{bmatrix}. \qquad (18.40)$$

Lemma 3.2. By a linear substitution of variables with constant coefficients, the system of differential equations (18.35) may be reduced to the form

$$\varepsilon \frac{dv_j}{d\tau} = f_j(v_1, v_2, \ldots, v_m; \tau, \varepsilon) \qquad (18.41)$$
$$(j = 1, 2, \ldots, m),$$

where $f_j(\nu_1, \nu_2, \ldots, \nu_m; \tau, \varepsilon)$ are polynomials of the second degree with respect to $\nu_1, \nu_2, \ldots, \nu_m$, with coefficients which depend on τ and ε, which can be expanded in power series of ε and which can be differentiated indefinitely with respect to τ. In this case the matrix of the principal linear part of system (18.41) has the form /98

$$P(\tau) = \begin{cases} \mu_j(\tau) & (j = k) \\ \delta_j & (j = k-1) \\ 0 & (j \neq k, k-1) \end{cases} \qquad (j, k = 1, 2, \ldots, m), \tag{18.42}$$

where

$$\mu_j(\tau) = \lambda_r(\tau) - \lambda_s(\tau) \qquad (r \neq s;\ r, s = 1, 2, \ldots, p;\ j = 1, \ldots, m),$$

while the numbers δ_j (j=1, 2, ..., m) are equal either to zero or to the number δ which we can select as arbitrarily small. In addition to this, if $\delta_j = \delta$, then $\mu_{j-1}(\tau) = \mu_j(\tau)$.

We shall first carry out the proof for matrices (18.39)-(18.40), after which it can be easily transposed to the general case. As it applies to our example, the principal linear part (18.38) of system (18.35) will consist of the matrices

$$W_{10}(\tau)\widetilde{Q}_{12}(\tau, \varepsilon) - \widetilde{Q}_{12}(\tau, \varepsilon) W_{20}(\tau) \tag{18.43}$$

and

$$W_{20}(\tau)\widetilde{Q}_{21}(\tau, \varepsilon) - \widetilde{Q}_{21}(\tau, \varepsilon) W_{10}(\tau). \tag{18.44}$$

By carrying out the specified operations with matrices (18.39) and (18.40), we find that in this case the system (18.35) assumes the form

$$\varepsilon \frac{du}{d\tau} = R(\tau) u + \ldots, \tag{18.45}$$

where the series of dots designates terms which are proportional to $\varepsilon, \varepsilon^2$, etc., $u = (u_1, u_2, \ldots, u_{12})$, while the matrix of the principal linear part has the form

$$R(\tau) = \begin{bmatrix} R_1(\tau) & 0 \\ 0 & R_2(\tau) \end{bmatrix}, \tag{18.46}$$

where, in turn,

$$R_1 = \begin{bmatrix} \mu_1(\tau) & 1 & 0 & 0 & 0 & 0 \\ 0 & \mu_1(\tau) & 1 & 0 & 0 & 0 \\ 0 & 0 & \mu_1(\tau) & 0 & 0 & 0 \\ -1 & 0 & 0 & \mu_1(\tau) & 1 & 0 \\ 0 & -1 & 0 & 0 & \mu_1(\tau) & 1 \\ 0 & 0 & -1 & 0 & 0 & \mu_1(\tau) \end{bmatrix}, \tag{18.47}$$

$$R_2(\tau) = \begin{bmatrix} \mu_2(\tau) & 0 & 0 & 1 & 0 & 0 \\ -1 & \mu_2(\tau) & 0 & 0 & 1 & 0 \\ 0 & -1 & \mu_2(\tau) & 0 & 0 & 1 \\ 0 & 0 & 0 & \mu_2(\tau) & 0 & 0 \\ 0 & 0 & 0 & -1 & \mu_2(\tau) & 0 \\ 0 & 0 & 0 & 0 & -1 & \mu_2(\tau) \end{bmatrix}, \tag{18.48}$$

$$\mu_1(\tau) = \lambda_1(\tau) - \lambda_2(\tau), \qquad \mu_2(\tau) = \lambda_2(\tau) - \lambda_1(\tau), \tag{18.49}$$

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and the zeros in the second diagonal of matrix (18.46) represent zero matrices of the sixth order.

Matrices (18.47) and (18.48) have a rather simple structure. We can see directly that $R_1(\tau)$ contains the eigenvalue $\mu_1(\tau)$ of multiplicity six, while $R_2(\tau)$ contains an eigenvalue $\mu_2(\tau)$ of the same multiplicity.

The fact that the eigenvalues of matrices $R_1(\tau)$ and $R_2(\tau)$ lie on the principal diagonal, while the nondiagonal elements do not depend on τ, is also true for the general case of system (18.35). It is precisely this situation which makes it possible to reduce system (18.35), by the linear substitution of variables with constant coefficients, to the form (18.41) with a matrix $P(\tau)$ for the principal linear part (see (18.42)). Let us show this by means of our example. Let us carry out the linear exchange of variables in system (18.45):

$$u = S\tilde{u}, \tag{18.50}$$

where the constant matrix S has a very simple structure:

$$S = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}, \tag{18.51}$$

$$S_1 = \begin{bmatrix} -\frac{1}{3} & 0 & 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -3 & 0 & 0 & 0 \\ -\frac{1}{3} & 0 & 0 & -2 & 0 & 0 \\ 0 & -\frac{2}{3} & 0 & 0 & -1 & 0 \end{bmatrix}, \tag{18.52}$$

$$S_2 = \begin{bmatrix} 0 & \frac{2}{3} & 0 & 0 & 1 & 0 \\ -\frac{1}{3} & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{1}{3} & 0 & 0 & -1 & 0 \\ -\frac{1}{3} & 0 & 0 & 1 & 0 & 0 \end{bmatrix}. \tag{18.53}$$

After this substitution system (18.45) will have the following form: /100

$$\varepsilon \frac{d\tilde{u}}{d\tau} = S^{-1}R(\tau)S\tilde{u} + \ldots. \tag{18.54}$$

We leave it up to the reader to become convinced that the matrix $S^{-1}R(\tau)S$ has the canonical Jordan form.

Finally let us carry out the linear transformation in system (18.54)

$$\tilde{u} = Dv, \qquad v = (\{v_1, v_2, \ldots, v_m\}) \tag{18.55}$$

with the matrix

$$D = \left[\begin{array}{cc|} D_1 & 0 \\ 0 & D_1 \end{array}\right|; \tag{18.56}$$

$$D_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta & 0 & 0 & 0 & 0 \\ 0 & 0 & \delta^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta^4 & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta^5 \end{bmatrix}, \tag{18.57}$$

where $\delta \neq 0$ is some number. Then system (18.54) will be transformed into the following:

$$\varepsilon \frac{dv}{d\tau} = D^{-1}S^{-1}R(\tau)SDv + \dots . \tag{18.58}$$

We introduce the designation

$$D^{-1}S^{-1}R(\tau)SD = P(\tau). \tag{18.59}$$

Thus, by a linear substitution of variables specified by the constant matrix SD (see equations (18.51) (18.57)), the system of differential equations (18.35), which corresponds to matrices (18.39) and (18.40), is transformed into the system /101

$$\varepsilon \frac{dv}{d\tau} = P(\tau)v + \dots \tag{18.60}$$

whose matrix for the principal linear part has the form

$$P(\tau) = \begin{bmatrix} P_1(\tau) & 0 \\ 0 & P_2(\tau) \end{bmatrix}, \tag{18.61}$$

where

$$P_1(\tau) = \begin{bmatrix} \mu_1(\tau) & \delta & 0 & 0 & 0 & 0 \\ 0 & \mu_1(\tau) & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_1(\tau) & \delta & 0 & 0 \\ 0 & 0 & 0 & \mu_1(\tau) & \delta & 0 \\ 0 & 0 & 0 & 0 & \mu_1(\tau) & \delta \\ 0 & 0 & 0 & 0 & 0 & \mu_1(\tau) \end{bmatrix}. \tag{18.62}$$

The matrix $P_2(\tau)$ differs from the matrix $P_1(\tau)$ only in the principal diagonal where in place of $\mu_1(t)$ it has $\mu_2(\tau)$. The zeros in the right side of (18.61), as in equations (18.51) and (18.46), designate the zero matrices of the corresponding order.

The remark made above concerning the general nature of the linear transformations that have been carried out here can be transposed directly to system (18.35) and make it possible to assume that Lemma 3.2 has been proven.

We emphasize the fact that the eigenvalues of the matrix for the principal linear part of system (18.35) are equal to various differences (18.43) obtained from pairs of different eigenvalues $\lambda_s(\tau)$(s=1, 2, ..., p); therefore the values $\mu_j(\tau)$(j=1, 2, ..., m) will not become equal to zero at any point of $\tau\in[0, L]$.

We also note that system (18.35) is equivalent to system (18.41) in the sense that for any solution of system (18.35) there is some solution of system (18.41) and vice versa. If we know, for example, the solution of system (18.35), we can indicate a corresponding solution of system (18.41). In particular, the formal solution (18.36) of system (18.35) is associated with some formal solution of system (18.41). Therefore if we can prove the asymptotic convergence of the formal solution for system (18.41), which corresponds to the formal solution (18.36) of system (18.35), then this will be equivalent to the proof of the asymptotic convergence of the formal solution (18.36).

We shall now proceed with the proof of this proposition.

7. Theorem 3.4. Let us assume that we are given a system of differential equations (18.41) which satisfies the conditions of Lemma 3.2 and we let

$$\mu_j(\tau) \neq 0 \qquad (j = 1, 2, \ldots, m), \qquad 0 \leqslant \tau \leqslant L. \tag{18.63}$$ /102

Let us assume that

$$v_j(\tau, \varepsilon) = \sum_{s=1}^{\infty} p_{js}(\tau)\varepsilon^s \qquad (j = 1, 2, \ldots, m) \tag{18.64}$$

is a formal solution of system (18.41) with coefficients (j=1, 2, ..., m; s=1, 2, ...), infinitely differentiable with respect to τ.

Then we can find a $\varepsilon_1 \leq \varepsilon_0$ such that there exists in the domain

$$0 \leqslant \tau \leqslant L, \qquad 0 < \varepsilon \leqslant \varepsilon_1 \qquad (\varepsilon_1 \leqslant \varepsilon_0) \tag{18.65}$$

a unique solution of system (18.41)

$$v_j = p_j(\tau, \varepsilon) \qquad (j = 1, 2, \ldots, m), \tag{18.66}$$

which is at least continuous with respect to τ, while the series (18.64) are the asymptotic expansions of solution (18.66) in the domain (18.65).

Before proceeding with the proof of this theorem we note that $\mu_j(\tau)$

(j=1, 2, ..., m), generally speaking, are complex functions of τ and we can select such complex constants $c_j=c_j'+ic_j''$ (j=1, 2, ..., m) such that the quantities

$$\sup_{\tau\in[0,L_1]} |\mu_j(\tau)-c_j| \qquad (j=1,2,\ldots,m) \tag{18.67}$$

will be arbitrarily small if the domain of approximation

$$0\leqslant\tau\leqslant L_1 \qquad (L_1\leqslant L) \tag{18.68}$$

is sufficiently small.

Moreover we can approach the functions $\mu_j(\tau)$ (j=1, 2, ..., m) on [0,L] by certain piece-wise constant functions of the form

$$C_j(\tau)=\begin{cases} c_{j0}=c_j, & 0\leqslant\tau\leqslant L_1,\\ c_{j1}, & L_1<\tau\leqslant L_2,\\ \cdots\cdots\cdots\cdots\cdots\\ c_{j,q-1}, & L_{q-1}<\tau\leqslant L_q=L\\ & (j=1,2,\ldots,m). \end{cases} \tag{18.69}$$

This partition of the interval [0,L] and this approximation of functions $\mu_j(\tau)$ (j=1, 2, ..., m) by piece-wise constant functions (18.69) is entirely feasible for any continuous function. However, since we know that $\mu_j(\tau)$ are not only continuous but also infinitely differentiable, they can be approximated quite well by piece-wise constant functions. /103

Let us first consider the functions $\mu_j(\tau)$ (j=1, 2, ..., m) in the first interval of approximation $0\leq\tau\leq L_1$. With slight changes, all the discussions can then be extended to the subsequent intervals $[L_i,L_{i+1}]$ (i=0, 1, 2, ..., q-1) which make up the entire interval [0,L].

Without sacrificing generality, we can propose that the first m' (m'<m) of functions $\mu_j(\tau)$ (j=1, 2, ..., m') have a nonnegative real part in the interval $[0,L_1]$ while the remaining functions $\mu_j(\tau)$ (j=m'+1, m'+2, ..., m) have a negative real part. Also, in selecting the interval $[0,L_1]$ we establish a condition that the real part of each of the functions $\mu_j(\tau)$ (j=1, 2, ..., m) inside the interval $[0,L_1]$ does not change its sign. In connection with this we should select the division points L_1, L_2, ..., L_{q-1}, L in equation (18.69) such that the values

$\tau\in[0,L]$ for which any of the functions $\mu_j(\tau)(j=1, 2, \ldots, m)$ changes the sign of the real part be contained in the set of separation points $L_i(i=1, 2, \ldots, p)$. Thus the number p will always be greater than or equal to the number of sign changes in functions Re $\mu_j(\tau)(j=1, 2, \ldots, m)$.

We note that among the first m' functions $\mu_j(\tau)(j=1, 2, \ldots, m')$ we may encounter those for which Re $\mu_j(\tau)\equiv 0(j=1, 2, \ldots, l, l\leq m')$.

Lemma 3.3 If τ and ε vary in the domain

$$0 \leqslant \tau \leqslant L_1, \qquad 0 < \varepsilon \leqslant \varepsilon_0 \qquad (L_1 \leqslant L), \tag{18.70}$$

then we can always find a positive constant c which is independent of ε such that the following inequalities will exist:

$$\left| \int_{\tau_j}^{\tau} \exp\left(-\frac{c_j(\tau-\tau_j)}{\varepsilon}\right) d\tau \right| \leqslant c\varepsilon \left| \exp\left(-\frac{c_j(\tau-\tau_j)}{\varepsilon}\right) \right|, \tag{18.71}$$

where

$$\tau_j = \begin{cases} L_1 \text{ for } j = 1, 2, \ldots, m'; \ c_j' \geqslant 0; \\ 0 \text{ for } j = m'+1, \ m'+2, \ldots, m; \ c_j' < 0. \end{cases} \tag{18.72}$$

Proof of the lemma becomes obvious on the basis of the following sequence of inequalities:

$$\begin{aligned} &\left| \int_{\tau_j}^{\tau} \exp\left(-\frac{c_j(\tau-\tau_i)}{\varepsilon}\right) d\tau \cdot \ \varepsilon \left| \frac{1}{c_j}\left[1 - \exp\left(-\frac{c_j(\tau-\tau_j)}{\varepsilon}\right)\right] \right| = \right. \\ &= \varepsilon \left| \frac{1}{c_j}\left[\exp\left(\frac{c_j(\tau-\tau_j)}{\varepsilon}\right) - 1\right] \right| \left| \exp\left(-\frac{c_j(\tau-\tau_j)}{\varepsilon}\right) \right| \leqslant \\ &\leqslant c\varepsilon \left| \exp\left(-\frac{c_j(\tau-\tau_j)}{\varepsilon}\right) \right| \qquad (j = 1, 2, \ldots, m), \end{aligned} \tag{18.73}$$

where

$$c = \max_{j=1,\ldots,m} \left\{ \frac{1}{|c_j|} \sup_{\substack{\tau\in[0,L_2] \\ 0<\varepsilon\leqslant\varepsilon_0}} \left| \exp\left(\frac{c_j(\tau-\tau_j)}{\varepsilon}\right) - 1 \right| \right\} = \max_{j=1,\ldots,m} \left\{ \frac{2}{|c_j|} \right\}. \tag{18.74}$$

In proving the theorem we shall be interested in those parts of the functions $f_j(\nu;\tau,\varepsilon)(j=1, 2, \ldots, m)$ which are linear with respect to ν in system (18.41):

$$a_j(\tau, \varepsilon) + \sum_{k=1}^{m} a_{jk}(\tau, \varepsilon) v_k \qquad (j = 1, 2, \ldots, m). \tag{18.75}$$

From the condition of the theorem, the coefficients of ν_j(j=1, ..., m) is partitioned into asymptotic series:

$$a_{jk}(\tau, \varepsilon) = \sum_{s=0}^{\infty} a_{jks}(\tau) \varepsilon^s \qquad (j = 1, 2, \ldots, m). \tag{18.76}$$

Then the first terms of expansions (18.76) which do not contain ε form the matrix $P(\tau)=[\alpha_{jk0}(\tau)]_{j,k=1}^{j,k=m}$, which has the form (18.42).

8. Proof of Theorem 3.4. Let us change variables in system (18.41)

$$v_j = p_j^{(N)}(\tau, \varepsilon) + y_j \qquad (j = 1, 2, \ldots, m), \tag{18.77}$$

where

$$p_j^{(N)}(\tau, \varepsilon) = \sum_{s=1}^{N-1} p_{js}(\tau) \varepsilon^s$$
$$(j = 1, 2, \ldots, m;\ N > 1), \tag{18.78}$$

i.e., the partial sum of series (18.64).

As a result of this change, system (18.41) will assume the form

$$\frac{dy_j}{dt} = g_j^{(N)}(y_1, y_2, \ldots, y_m; \tau, \varepsilon)$$
$$(j = 1, 2, \ldots, m), \tag{18.79}$$

where the right sides as before are polynomials in y_j(j=1, 2, ..., m) of the second degree with coefficients which are infinitely differentiable with respect to τ, and these polynomials arrange themselves in the asymptotic series according to the powers of parameter ε in the domain $0 \le \tau \le L, 0 < \varepsilon \le \varepsilon_0$.

From an analogy with (18.75), the linear parts $g_j^{(N)}(y;\tau,\varepsilon)$ have the form /105

$$b_j^{(N)}(\tau, \varepsilon) + \sum_{k=1}^{m} b_{jk}^{(N)}(\tau, \varepsilon) y_k$$
$$(j = 1, 2, \ldots, m). \tag{18.80}$$

Also there exist expansions of coefficients

$$b_{jk}^{(N)}(\tau, \varepsilon) = \sum_{s=0}^{\infty} b_{jks}^{(N)}(\tau)\,\varepsilon^s \qquad (j, k = 1, 2, \ldots, m). \tag{18.81}$$

Then, by a direct substitution of (18.77) into (18.75) and (18.41) we can easily see that the matrix for the coefficients of the principal linear part of system (18.79) coincides with (18.42), i.e.,

$$b_{jk0}^{(N)}(\tau) = a_{jk0}(\tau) = \begin{cases} \mu_j(\tau) & (k = j), \\ \delta_j & (k = j+1), \\ 0 & (k \neq j, j+1). \end{cases} \tag{18.82}$$

On the other hand, system (18.79) has a formal solution

$$y_j = \sum_{s=N}^{\infty} p_{js}(\tau)\,\varepsilon^s \qquad (j = 1, 2, \ldots, m), \tag{18.83}$$

which follows from the substitution (18.77) and from the conditions of the theorem.

Therefore the expansions of free terms $b_j^{(N)}(\tau, \varepsilon)$ $(j=1, 2, \ldots, m)$ do not contain powers of ε below N. Consequently we can find a constant B_N so that the following inequality is satisfied:

$$|b_j^{(N)}(\tau, \varepsilon)| \leqslant B_N \varepsilon^N \qquad (j = 1, 2, \ldots, m) \tag{18.84}$$

for $0 \leq \tau \leq L$, $0 < \varepsilon \leq \varepsilon_0$, and B_N does not depend on ε. We let

$$\|y\| = \max\{|y_1|, |y_2|, \ldots, |y_m|\}. \tag{18.85}$$

We note that δ in (18.42) may be selected so that it is arbitrarily small. Then we specify such $\delta, \varepsilon_1 \leq \varepsilon_0$ and $L_1 \leq L$, so that the following inequalities are satisfied:

$$|b_{jj}^{(N)}(\tau, \varepsilon) - c_j| \leqslant A_1 \qquad (j = 1, 2, \ldots, m), \tag{18.86}$$

$$|b_{jk}^{(N)}(\tau, \varepsilon)| \leqslant A_1 \qquad (j \neq k;\ j, k = 1, 2, \ldots, m), \tag{18.87}$$

if

$$0 \leqslant \tau \leqslant L_1, \qquad 0 < \varepsilon \leqslant \varepsilon_1. \tag{18.88}$$

The constant A_1 may be selected sufficiently small if ε, δ and L_1 are also sufficiently small. We shall return to the question of selecting these constants.

Let us replace the variables in system (18.79) by letting

$$y_j = z_j \exp\left[\frac{c_j(\tau - \tau_j)}{\varepsilon}\right] \qquad (j = 1, 2, \ldots, m). \tag{18.89}$$

Then from system (18.79) we obtain a system of equations with respect to z_j (j=1, 2, ..., m):

$$\begin{aligned}\varepsilon \frac{dz_j}{d\tau} = g_j^{(N)}\left[z_1 \exp\left(\frac{c_1(\tau-\tau_1)}{\varepsilon}\right), \ldots, z_m \exp\left(\frac{c_m(\tau-\tau_m)}{\varepsilon}\right); \tau, \varepsilon\right] \times \\ \times \exp\left(-\frac{c_j(\tau-\tau_j)}{\varepsilon}\right) - c_j z_j \qquad (j = 1, 2, \ldots, m).\end{aligned} \tag{18.90}$$

To evaluate the right sides of system (18.90) it is convenient to retain partially the old variables y_j (j=1, 2, ..., m) in system (18.90). Comparing (18.89) and (18.90), it is easy to see that system (18.90) may be written in the form

$$\begin{gathered}\varepsilon \frac{dz_j}{d\tau} = \{g_j^{(N)}(y_1, \ldots, y_m; \tau, \varepsilon) - c_j y_j\} \exp\left(-\frac{c_j(\tau - \tau_j)}{\varepsilon}\right) \\ (j = 1, 2, \ldots, m),\end{gathered} \tag{18.91}$$

where τ_j are selected in accordance with (18.72).

Let us evaluate the expressions in system (18.91) which are contained in the braces.

Functions $g_j^{(N)}(y;\tau,\varepsilon)$ (j=1, ..., m) consist of linear terms of form (18.80) and nonlinear parts whose structure may be clarified if we examine the right sides of system (18.35) from which they are obtained after the linear substitution of variables. We have already pointed out above that the coefficients of the nonlinear terms in system (18.35) are infinitesimals of order ε. We can say the same about the right sides of system (18.79). Then, by taking into account inequalities (18.84), (18.86) and (18.87), we obtain

$$\begin{gathered}|g_j^{(N)}(y; \tau, \varepsilon) - c_j y_j| \leqslant B_N \varepsilon^N + A_1 m \|y\| + \varepsilon H m^2 \|y\|^2 \\ (j = 1, 2, \ldots, m),\end{gathered} \tag{18.92}$$

where the first term in the right side corresponds to estimate (18.84) and the second term to estimates (18.86) and (18.87), while the constant H is used

to evaluate the maximum coefficients of the nonlinear (quadratic with respect to y) terms in system (18.79). When ϵ and $\|y\|$ are sufficiently small, the third term on the right side of inequality (18.92) may be made arbitrarily small. We write /107

$$A_1 m \|y\| + \varepsilon H m^2 \|y\|^2 = m\|y\|(A_1 + \varepsilon H m \|y\|) = Am\|y\|, \tag{18.93}$$

where

$$A = A_1 + \varepsilon H m \|y\|. \tag{18.84}$$

Consequently,

$$|g_j^{(N)}(y; \tau, \varepsilon) - c_j y_j| \leqslant B_N \varepsilon^N + Am\|y\| \tag{18.95}$$
$$(j = 1, 2, \ldots, m)$$

when

$$0 \leqslant \tau \leqslant L_1, \qquad 0 < \varepsilon \leqslant \varepsilon_1.$$

We shall select L_1, δ and ε_1 sufficiently small so that the following inequality will be satisfied

$$Amc < 1, \tag{18.96}$$

where c is determined by means of equation (18.74). Inequalities (18.95) and (18.96) will be required in the future.

Let $\psi_j(\varepsilon)$ (j=1, 2, ..., m) be some functions of $\varepsilon (0 < \varepsilon \leq \varepsilon_0)$ which are expanded in power series of ε:

$$\psi_j(\varepsilon) = \sum_{s=1}^{\infty} p_{js}(\tau_j)\varepsilon^s \qquad (j = 1, 2, \ldots, m). \tag{18.97}$$

We let

$$\psi_j^{(N)}(\varepsilon) = \psi_j(\varepsilon) - p_j^{(N)}(\tau_j, \varepsilon) \qquad (j = 1, 2, \ldots, m), \tag{18.98}$$

where τ_j are constants determined from (18.72), while $p_j^{(N)}(\tau_j, \varepsilon)$ are determined from (18.78).

Then the following inequality exists

$$|\psi_j^{(N)}(\varepsilon)| \leqslant R_N \varepsilon^N \qquad (j = 1, 2, \ldots, m),$$
$$|\psi_j^{(N)}(\varepsilon_2) - \psi_j^{(N)}(\varepsilon_1')| \leqslant k_N |\varepsilon_2 - \varepsilon_1'| \varepsilon^{N-1} \quad (\varepsilon_1', \varepsilon_2 \in [0, \varepsilon_1]), \tag{18.99}$$

which follows from the differentiability of $p_j(\tau, \varepsilon)(j=1, \ldots, m)$ with respect to ε, where R_N and k_N are constants which are independent of ε and which are common for all $j=1, 2, \ldots, m$.

Let us prove the existence and uniqueness of solution

$$z_j = \psi_j^{(N)}(\tau, \varepsilon) \qquad (j = 1, 2, \ldots, m) \cdot \tag{18.100}$$

of system (18.90), continuous with respect to τ, which satisfies conditions /108

$$\psi_j^{(N)}(\tau_j, \varepsilon) = \psi_j^{(N)}(\varepsilon) \qquad (j = 1, 2, \ldots, m) \tag{18.101}$$

and inequalities

$$|\psi_j^{(N)}(\tau, \varepsilon)| \leqslant K_N \varepsilon^N \left| \exp\left(-\frac{c_j(\tau - \tau_j)}{\varepsilon}\right)\right| \tag{18.102}$$
$$(j = 1, 2, \ldots, m)$$

in the domain (18.88): $0 \leq \tau \leq L_1, 0 < \varepsilon \leq \varepsilon_1$. The constant K_N is independent of ε and, as we shall show below, may be determined in the following manner:

$$K_N = \frac{R_N + cB_N}{1 - Amc}. \tag{18.103}$$

The proof of the existence and uniqueness of solution (18.100) of system (18.90) which satisfies conditions (18.101) and (18.102), in view of (18.89), is equivalent to the proof of the existence and uniqueness of the solution of (18.79), which satisfies the conditions

$$|y_j(\tau, \varepsilon)| \leqslant K_N \varepsilon^N \qquad (j = 1, 2, \ldots, m) \tag{18.104}$$

and analogous conditions (18.101). However, if we take into account equations (18.77), (18.78) and (18.64), it then becomes clear that the proof of this proposition concerning solution (18.100) will be equivalent to the proof of Theorem 3.4 for the domain (18.88).

In essence, the proof is reduced to the application of the principle of a stationary point for Banach spaces (the Shauder principle (ref. 33)). We can also make reference to the theory of analogous content, which is formulated for linear topological spaces in reference 135. In this case we shall use the concepts of convexity and compactness as they apply to sets in the Banach (or topological) spaces. The reader may find the existing determinations and other necessary information, for example, in the book by L. D. Kantorovich and G. P. Akilov (ref. 33).

Continuing with the proof of Theorem 3.4, we shall consider the set

Φ_j (j=1, 2, ..., m) of functions $\varphi_j(\tau, \varepsilon)$ determined in the domain (18.88), which are infinitely differentiable with respect to τ, which have a continuous derivative with respect to ε and which satisfy the following inequalities in the specified domain:

$$|\varphi_j(\tau, \varepsilon)| \leqslant K_N \varepsilon^N \left| \exp\left(-\frac{c_j(\tau - \tau_j)}{\varepsilon} \right) \right|, \tag{18.105}$$

$$\left| \varphi_j(\tau, \varepsilon_2) \exp\left(\frac{c_j(\tau - \tau_j)}{\varepsilon_2} \right) - \varphi_j(\tau, \varepsilon_1) \exp\left(\frac{c_j(\tau - \tau_j)}{\varepsilon_1} \right) \right| \leqslant \tag{18.106}$$

$$\leqslant h_N \varepsilon^{N-2} |\varepsilon_2 - \varepsilon_1'|,$$

$$\left| \varphi_j(\tau_2, \varepsilon) \exp\left(\frac{c_j(\tau_2 - \tau_j)}{\varepsilon} \right) - \varphi_j(\tau_1, \varepsilon) \exp\left(\frac{c_j(\tau - \tau_j)}{\varepsilon} \right) \right| \leqslant$$

$$\leqslant M_N \varepsilon^{N-1} |\tau_2 - \tau_1| \tag{18.107}$$

$$(j = 1, 2, \ldots, m;\ \tau_1, \tau_2 \in [0, L_1],\ 0 < \varepsilon \leqslant \varepsilon_1),$$

where the constant K_n was determined above by means of equation (18.103), while h_N and M_N are certain constants which do not depend on ε. In particular, we may let

$$M_N = R_N c^* + (1 + c^* c)(B_N + A_m K_N), \tag{18.108}$$

where

$$c^* = \max_{1 \leqslant j \leqslant m} |c_j|. \tag{18.109}$$

To each set Φ_j (j=1, 2, ..., m) there exists a corresponding set F_j (j=1, 2, ..., m) of functions $f_j(\tau, \varepsilon)$ which are assigned in the same region (18.88) and which are associated with the functions $\varphi_j \in \Phi_j$ by the relationship

$$f_j(\tau, \varepsilon) = \varphi_j(\tau, \varepsilon) \exp\left(\frac{c_j(\tau - \tau_j)}{\varepsilon} \right) \tag{18.110}$$

$$(j = 1, 2, \ldots, m).$$

Equalities (18.110) obviously correspond to the replacement of variables

in (18.89) and establish a unique mutual relationship between the elements of the sets Φ_j and $F_j (j=1, \ldots, m)$.

Then the following inequalities for functions $f_j(\tau, \varepsilon) \in F_j (j=1, 2, \ldots, m)$ follow directly from inequalities (18.105)-(18.107) in accordance with (18.110):

$$|f_j(\tau, \varepsilon)| \leqslant K_N \varepsilon^N, \tag{18.111}$$

$$|f_j(\tau, \varepsilon_2) - f_j(\tau, \varepsilon_1')| \leqslant h_N \varepsilon^{N-2} |\varepsilon_2 - \varepsilon_1'|, \tag{18.112}$$

$$|f_j(\tau_2, \varepsilon) - f_j(\tau_1, \varepsilon)| \leqslant M_N \varepsilon^{N-1} |\tau_2 - \tau_1|$$

$$(j = 1, 2, \ldots, m;\ \tau_1, \tau_2 \in [0, L],\ 0 < \varepsilon \leqslant \varepsilon_1), \tag{18.113}$$

where

$$0 < \varepsilon_1' \leqslant \varepsilon_2 \leqslant \varepsilon_1. \tag{18.114}$$

Let us introduce in sets $F_j (j=1, 2, \ldots, m)$ a topology of uniform convergence, i.e., a topology of space C of continuous functions. We shall assume that two functions $f_j^{(1)}, f_j^{(2)}, \in f_j$ are sufficiently close if the "distance" ρ between them, which is determined from the equations /110

$$\rho(f_j^{(1)}, f_j^{(2)}) = \max_{\substack{\tau \in [0, L_1] \\ 0 < \varepsilon \leqslant \varepsilon_1}} |f_j^{(1)}(\tau, \varepsilon) - f_j^{(2)}(\tau, \varepsilon)| \tag{18.115}$$

$$(j = 1, 2, \ldots, m),$$

is sufficiently small. Thus we can assume that $F_j \subset C$. We shall show that inequalities (18.111)-(18.113) provide for the compactness of sets $F_j (j=1, 2, \ldots, m)$.

According to the Artsel Theorem (ref. 33), the set of functions in space C is compact if the functions of this set are uniformly bounded and continuous in equal degrees. The fact that the functions from F_j are uniformly bounded follows from inequality (18.111). The requirement of continuity in equal degree may be replaced by the Lipschitz condition (see reference 33)). Thus we must show that functions $f_j \in F_j$ satisfy the Lipschitz condition with respect to τ and ε in the domain (18.88):

$$|f_j(\tau_2, \varepsilon_2) - f_j(\tau_1, \varepsilon_1')| \leqslant k_1|\tau_2 - \tau_1| + k_2|\varepsilon_2 - \varepsilon_1'| \quad (j = 1, 2, \ldots, m), \tag{18.116}$$

where τ_1, τ_2, ε_1' and ε_2 are any values of τ and ε from (18.88), while k_1 and k_2 are constants which are independent of τ and ε. A stronger inequality, compared with (18.116), follows from inequalities (18.112) and (18.113).

Indeed

$$|f_j(\tau_2, \varepsilon_2) - f_j(\tau_1, \varepsilon_1')| = |f_j(\tau_2, \varepsilon_2) - f_j(\tau_1, \varepsilon_2) + f_j(\tau_1, \varepsilon_2) - f_j(\tau, \varepsilon_1')| \leqslant |f_j(\tau_2, \varepsilon_2) - f_j(\tau_1, \varepsilon_2)| + |f_j(\tau_1, \varepsilon_2) - f_j(\tau_1, \varepsilon_1')| \quad (j = 1, 2, \ldots, m;\ \tau_1, \tau_2 \in [0, L_1];\ \varepsilon_1', \varepsilon_2 \in (0, \varepsilon_1]). \tag{18.117}$$

In evaluating the first term on the right side of (18.117) we refer to inequality (18.113), and when evaluating the second term, we make use of estimate (18.112). Then we obtain

$$|f_j(\tau_2, \varepsilon_2) - f_j(\tau_1, \varepsilon_1')| \leqslant M_N\varepsilon^{N-1}|\tau_2 - \tau_1| + h_N\varepsilon^{N-2}|\varepsilon_2 - \varepsilon_1'| \quad (j = 1, 2, , \ldots, m;\ 0 < \varepsilon_1' \leqslant \varepsilon_2 \leqslant \varepsilon_1;\ \varepsilon_1' \leqslant \varepsilon \leqslant \varepsilon_2). \tag{18.118}$$

Obviously inequality (18.118) is even stronger than (18.116). Thus we have shown that, on the basis of (18.111) and (18.118), the sets F_j (j=1, ..., m) are compact in the metrics of space C of continuous functions. However, in this case the sets Φ_j (j=1, 2, ..., m) are also compact in the metrics which are assigned in accordance with (18.115) and (18.110) by the following equalities

/111

$$\tilde{\varrho}_j(\varphi_j^{(1)}, \varphi_j^{(2)}) = \max_{\substack{\tau\in[0,L_1]\\ 0<\varepsilon\leqslant\varepsilon_1}} \left|\exp\left(\frac{c_j(\tau - \tau_j)}{\varepsilon}\right)[\varphi_j^{(1)}(\tau, \varepsilon) - \varphi_j^{(2)}(\tau, \varepsilon)]\right| \quad (j = 1, 2, \ldots, m). \tag{18.119}$$

We note that sets Φ_j (j=1, 2, ..., m) are convex. This directly follows inequalities (18.105)-(18.107). Indeed, let us take two positive numbers α and β such that $\alpha+\beta=1$, and any two functions $\varphi_j^{(1)}, \varphi_j^{(2)} \in \Phi_j$. Then it follows from inequality (18.105) that

$$|\alpha\varphi_j^{(1)} + \beta\varphi_j^{(2)}| \leqslant \alpha|\varphi_j^{(1)}| + \beta|\varphi_j^{(2)}| \leqslant$$

$$\leqslant (\alpha + \beta) K_N \varepsilon^N \left| \exp\left(-\frac{c_j(\tau - \tau_j)}{\varepsilon}\right)\right| = K_N \varepsilon^N \left| \exp\left(-\frac{c_j(\tau - \tau_j)}{\varepsilon}\right)\right| \qquad (18.120)$$

$$(j = 1, 2, \ldots, m).$$

In exactly the same way we can show that the function $\varphi_j^{(3)}(\tau, \varepsilon) = \alpha\varphi_j^{(1)}(\tau, \varepsilon) + \beta\varphi_j^{(2)}(\tau, \varepsilon)$ also satisfies inequalities (18.106) and (18.107). Consequently, $\varphi_j^{(3)}(\tau, \varepsilon) \in \Phi_j$. However, this means that the sets Φ_j (j=1, 2, ..., m) are convex.

Let us consider the direct product of sets Φ_j (j=1, ..., m):

$$F = \Phi_1 \times \Phi_2 \times \ldots \times \Phi_m. \qquad (18.121)$$

The elements of the set F will be variously ordered collections of m functions: $\{\varphi_1(\tau, \varepsilon), \ldots, \varphi_m(\tau, \varepsilon)\}$, where $\varphi_j(\tau, \varepsilon) \in \Phi_j$ (j=1, 2, ..., m).

From the convexity and compactness of sets Φ_j (j=1, 2, ..., m) will follow the convexity and compactness of sets F in the metrics assigned by the collection of distances (18.119). This follows from the properties of the direct product of metric or topological spaces (ref. 33).

The system of differential equations (18.90) with conditions (18.101) may be written in the following integral form:

$$z_j(\tau, \varepsilon) = \psi_j^{(N)}(\varepsilon) + \frac{1}{\varepsilon}\int_{\tau_j}^{\tau} G_j(z_1, \ldots, z_m; \tau, \varepsilon)\, d\tau \qquad (18.122)$$

$$(j = 1, 2, \ldots, m),$$

where $G_j(z; \tau, \varepsilon)$ designate the right sides of equations (18.90). Proceeding /112 from system (18.122), we can assume that the equality

$$\Psi_j(\tau, \varepsilon) = \psi_j^{(N)}(\varepsilon) + \frac{1}{\varepsilon}\int_{\tau_j}^{\tau} G_j(\varphi_1, \ldots, \varphi_m; \tau, \varepsilon)\, d\tau \qquad (18.123)$$

$$(j = 1, 2, \ldots, m)$$

can be used to assign some operator T in the set F. This operator associates an entirely new, fully determined collection of functions $\{\varphi_1(\tau, \varepsilon), \varphi_2(\tau, \varepsilon) \ldots, \varphi_m(\tau, \varepsilon)\}$

with each element of the set F, i.e., with each ordered collection of functions

$$\{\Psi_1(\tau, \varepsilon), \ldots, \Psi_m(\tau, \varepsilon)\}.$$

We recall that $G_j(\varphi_1, \ldots, \varphi_m; \tau, \varepsilon)$ is a polynomial of second order with respect to $\varphi_1, \ldots, \varphi_m$ with infinitely smooth coefficients with respect to τ which has been multiplied by $\exp\left(-\frac{c_j(\tau-\tau_j)}{\varepsilon}\right)$. Then from the form of the right side of (18.123) it follows that the operator T reduces the functions which are infinitely differentiable with respect to τ into infinitely differentiable functions. It is easy to show that T is a continuous operator, i.e., that T transforms the close elements into other close elements (in the assumed metrics).

Now we must show that the operator T transforms the set F into itself, i.e., we must prove the inclusion

$$\tilde{T}\{F\} \subseteq F. \tag{18.124}$$

This means that the functions $\Psi_j(\tau, \varepsilon)$ established from (18.123) must satisfy inequalities (18.105)-(18.107). If we can show this, the compactness of $T\{F\}$ and the inclusion (18.124) will thereby be proved.

Let us prove estimates of type (18.105) for $\Psi_j(\tau, \varepsilon)$ (j=1, 2, ..., m). We note that inequalities (18.111) may be combined into a single inequality

$$\|f\| \leqslant K_N \varepsilon^N, \tag{18.125}$$

if we let

$$\|f\| = \max\{|f_1(\tau, \varepsilon)|, \ldots, |f_m(\tau, \varepsilon)|\}. \tag{18.126}$$

Taking into consideration the association between systems (18.90) and 18.91) and also relationships (18.89) and (18.110), we can estimate the quanity (18.123) in the following manner:

$$|\Psi_j(\tau, \varepsilon)| \leqslant |\psi_j^{(N)}(\varepsilon)| + \frac{1}{\varepsilon}\left|\int_{\tau_j}^{\tau} \{g_j^{(N)}(f_1, \ldots, f_m; \tau, \varepsilon) - c_j f_j\} \times \exp\left(-\frac{c_j(\tau-\tau_j)}{\varepsilon}\right) d\tau\right| \quad (j = 1, 2, \ldots, m). \tag{18.127}$$

stimates (18.95) and (18.99) remain valid if we replace y_j with f_j(j=1, ...,) in them. This follows from (18.85), (18.126) and (18.125). If we make use f these estimates and of Lemma 3.3, we obtain the following from inequality 18.127): /113

$$
\begin{aligned}
|\Psi_j(\tau,\varepsilon)| &\leqslant R_N\varepsilon^N+\frac{1}{\varepsilon}(B_N\varepsilon^N+Am\|f\|)\left|\int_{\tau_j}^{\tau}\exp\left(-\frac{c_j(\tau-\tau_i)}{\varepsilon}\right)d\tau\right|< \\
&<R_N\varepsilon^N+(B_N\varepsilon^N+Am\|f\|)c\left|\exp\left(-\frac{c_j(\tau-\tau_i)}{\varepsilon}\right)\right|= \\
&=\varepsilon^N(R_N+cB_N)\left[1+\frac{Am\|f\|}{(R_N+B_Nc)\varepsilon^N}\right]\left|\exp\left(-\frac{c_j(\tau-\tau_j)}{\varepsilon}\right)\right|= \\
&=\varepsilon^N(R_N+cB_N)\left[1+\frac{Amc\|f\|}{(1-Amc)\dfrac{R_N+cB_N}{1-Amc}\varepsilon^N}\right]\left|\exp\left(-\frac{c_j(\tau-\tau_j)}{\varepsilon}\right)\right| \\
&\qquad (j=1,2,\ldots,m).
\end{aligned}
\tag{18.128}
$$

Making use of designation (18.103) and inequality (18.125), we find

$$
|\Psi_j(\tau,\varepsilon)|\leqslant K_N\varepsilon^N\left|\exp\left(\frac{-c_j(\tau-\tau_j)}{\varepsilon}\right)\right| \quad (j=1,2,\ldots,m), \tag{18.129}
$$

which is what was required.

To obtain inequalities of type (18.106) for the functions $\Psi_j(\tau,\varepsilon)$ (j=1, 2, ..., m) we introduce the designations

$$
\left|\Psi_j(\tau,\varepsilon_2)\exp\frac{c_j(\tau-\tau_j)}{\varepsilon_2}-\Psi_j(\tau,\varepsilon_1')\exp\frac{c_j(\tau-\tau_j)}{\varepsilon_1'}\right|=\Delta(\Psi_j,\varepsilon_1',\varepsilon_2) \tag{18.130}
$$
$$
(j=1,2,\ldots,m).
$$

Then, proceeding from equalities (18.123), we find

$$
\begin{aligned}
\Delta(\Psi_j,\varepsilon_1',\varepsilon_2)&\leqslant\left|\Psi_j^N(\varepsilon_2)\exp\frac{c_j(\tau-\tau_l)}{\varepsilon_2}-\Psi_j^{(N)}(\varepsilon_1')\exp\frac{c_j(\tau-\tau_j)}{\varepsilon}\right|+ \\
&+\left|\frac{1}{\varepsilon_2}\exp\left(\frac{c_j(\tau-\tau_l)}{\varepsilon_2}\right)\int_{\tau_j}^{\tau}G_j(\varphi;\tau,\varepsilon_2)\,d\tau-\frac{1}{\varepsilon_1'}\exp\left(\frac{c_j(\tau-\tau_j)}{\varepsilon}\right)\times\right. \\
&\left.\times\int_{\tau_j}^{\tau}G_j(\varphi;\tau,\varepsilon_1')\,d\tau\right| \qquad (j=1,2,\ldots,m).
\end{aligned}
\tag{18.131}
$$

To evaluate the first term on the right side of inequality (18.131) we can add and subtract expression $\psi_j^{(N)}(\varepsilon_1')\exp\frac{c_j(\tau-\tau_j)}{\varepsilon}$. Then from relationships (18.89) and (18.109) it is easy to obtain the inequality /114

$$\left|\psi_j^{(N)}(\varepsilon_2)\exp\frac{c_j(\tau-\tau_j)}{\varepsilon_2}-\psi_j^{(N)}(\varepsilon_1')\exp\frac{c_j(\tau-\tau_j)}{\varepsilon_1'}\right|\leqslant$$
$$\leqslant k_N'\varepsilon^{N-2}|\varepsilon_2-\varepsilon_1'| \qquad (j=1,2,\ldots,m;\ \varepsilon_1'\leqslant\varepsilon\leqslant\varepsilon_2), \tag{18.132}$$

where

$$k_N'=k_N\varepsilon_1+R_Nc^*. \tag{18.133}$$

For the sake of brevity, the second term on the right side of (18.131) will be designated by $\Delta_2(\Psi_j,\varepsilon_1',\varepsilon_2)$. By using the theorem on the average from inequality (18.131) and system (18.91) we obtain the following estimate for this term.

$$\Delta_2(\Psi_j,\varepsilon_1',\varepsilon_2)=\left|\frac{d}{d\varepsilon}\frac{1}{\varepsilon}\int_{\tau_j}^{\tau}\{g_j^{(N)}(f_1,\ldots,f_m;\tau,\varepsilon)-c_jf_j\}\,d\tau\right|_{\varepsilon=\varepsilon_j^*}|\varepsilon_2-\varepsilon_1'|, \tag{18.134}$$

where $\varepsilon_1'\leqslant\varepsilon_j^*\leqslant\varepsilon_2;\quad j=1,2,\ldots,m.$

In estimating the derivative of the integral term in (18.134), we use the fact that the functions $f_j(\tau,\varepsilon)$ and the coefficients of the initial system (18.90) are differentiable with respect to ε and the fact that inequalities of type (18.128) are repeated. As a result we obtain the estimate

$$\Delta_2(\Psi_j,\varepsilon_1',\varepsilon_2)\leqslant H_N\varepsilon^{N-2}|\varepsilon_2-\varepsilon_1'| \qquad (j=1,2,\ldots,m;\ \varepsilon_1'\leqslant\varepsilon\leqslant\varepsilon_2), \tag{18.135}$$

where the constant H_N is independent of ε.

Then, in order to represent (18.130), we use the following inequalities, which follow from inequalities (18.132) and (18.135):

$$\Delta(\Psi_j,\varepsilon_1',\varepsilon_2)\leqslant(H_N+k_N')\varepsilon^{N-2}|\varepsilon_2-\varepsilon_1'| \tag{18.136}$$
$$(j=1,2,\ldots,m;\quad \varepsilon\in[\varepsilon_1',\varepsilon_2]\subset(0,\varepsilon_1]).$$

Thus, if we set the constant h_N equal to $h_N=H_N+k_N'$ in inequality (18.106) until now h_N has remained arbitrary), then inequalities (18.106) will be satisfied not only by the functions $\varphi_j(\tau,\varepsilon)\in\Phi_j$, but also by the functions $\Psi_j(\tau,\varepsilon)$ (j=1, ..., m) constructed in accordance with equations (18.123).

It now remains for us to prove that conditions (18.107) are satisfied /115

for $\Psi_j(\tau, \varepsilon)$ (j=1, 2, ..., m). Substituting (18.123) into the left sides of (18.107) in place of $\varphi_j(\tau, \varepsilon)$ and carrying out the corresponding estimates, we obtain

$$\left|\Psi_j(\tau_2, \varepsilon)\exp\left(\frac{c_j(\tau_2-\tau_j)}{\varepsilon}\right)-\Psi_j(\tau_1, \varepsilon)\exp\left(\frac{c_j(\tau_1-\tau_j)}{\varepsilon}\right)\right|\leqslant$$

$$\leqslant\left|\psi_j^{(N)}(\varepsilon)\left[\exp\left(\frac{c_j(\tau_2-\tau_j)}{\varepsilon}\right)-\exp\left(\frac{c_j(\tau_1-\tau_j)}{\varepsilon}\right)\right]\right|+$$

$$+\frac{1}{\varepsilon}\left|\exp\left(\frac{c_j(\tau_2-\tau_j)}{\varepsilon}\right)\int_{\tau_j}^{\tau_2}G_j(\varphi;\tau,\varepsilon)\,d\tau-\exp\left(\frac{c_j(\tau_2-\tau_j)}{\varepsilon}\right)\int_{\tau_j}^{\tau_1}G_j(\varphi;\tau,\varepsilon)\,d\tau\right|$$

$$(j=1,\ldots,m;\ \tau_1,\tau_2\in[0,L_1],\ 0<\varepsilon\leqslant\varepsilon_1). \tag{18.137}$$

In view of (18.99), the first term on the right side of (18.137) is majored by the quantity $R_N c^*|\tau_2-\tau_1|\varepsilon^{N-1}$; $c^*=\max\limits_{1\leqslant j\leqslant m}|c_j|$. To evaluate the second term in inequality (18.137) we add and subtract the following quantity from it:

$$\exp\left(\frac{c_j(\tau-\tau_j)}{\varepsilon}\right)\int_{\tau_j}^{\tau_1}G_j(\varphi;\tau,\varepsilon)\,d\tau. \tag{18.138}$$

After this we make estimates analogous to (18.128). Dropping the intermediate steps, which can be carried out quite easily, we write the final result

$$\left|\Psi_j(\tau_2, \varepsilon)\exp\left(\frac{c_j(\tau_2-\tau_j)}{\varepsilon}\right)-\Psi_j(\tau_1, \varepsilon)\exp\left(\frac{c_j(\tau_1-\tau_j)}{\varepsilon}\right)\right|\leqslant$$

$$\leqslant[R_N c^*+(1+cc^*)(B_N+AmK_N)]\,\varepsilon^{N-1}|\tau_2-\tau_1|=M_N\varepsilon^{N-1}|\tau_2-\tau_1|$$

$$(j=1,2,\ldots,m), \tag{18.139}$$

where the constant M_N obviously coincides with the constant M_N in the right side of (18.107) and (18.108). Thus a comparison of inequalities (18.105), (18.106) and (18.107) with the respective inequalities (18.129), (18.136) and (18.139), and also the fact that the operator T determined above does not decrease the smoothness of functions, make it possible for us to conclude that the image $T\{F\}$ of the set F is contained within F, i.e., inclusion (18.124) is satisfied.

Now we need only refer to the Shauder principle (ref. 33) or to the theorem concerning invariant points which has been proven in the work of M. Hukukhara (135): if a continuous operator T transforms a convex compact set F into a subset of this set, i.e., if $T\{F\}\subseteq F$, then the set F contains at least one point x which is invariant with respect to the operator T and for which /116

$$Tx=x. \tag{18.140}$$

In other words, at least one collection of functions $\psi_1^{(N)}(\tau,\varepsilon)$, $\psi_2^{(N)}(\tau,\varepsilon),\ldots,$ $\psi_m^{(N)}(\tau,\varepsilon)$ from F exists, which transforms the system (18.122) into a system of identities

$$\psi_j^{(N)}(\tau,\varepsilon)\equiv\psi_j^{(N)}(\varepsilon)+\varepsilon^{-1}\int_{\tau_j}^{\tau}G_j(\psi_1^{(N)},\ldots,\psi_m^{(N)};\tau,\varepsilon)\,d\tau\quad(j=1,2,\ldots,m). \tag{18.141}$$

Consequently we have shown the existence of the solution of a system of differential equations (18.90) satisfying conditions (18.101) and (18.102). Let us prove the uniqueness of this solution. Let us assume that there exists another solution

$$z_j = \chi_j^{(N)}(\tau, \varepsilon) \qquad (j = 1, 2, \ldots, m), \tag{18.142}$$

which satisfies conditions of the form (18.101) and (18.102) and which differs from solution (18.100). We let

$$M(\varepsilon) = \max_{\tau, j} \left| [\chi_j^{(N)}(\tau, \varepsilon) - \psi_j^{(N)}(\tau, \varepsilon)] \exp \frac{c_j(\tau - \tau_j)}{\varepsilon} \right|. \tag{18.143}$$

Then, taking into account inequalities (18.86), (18.92) and (18.95), we obtain the following estimate for $M(\varepsilon)$:

$$M(\varepsilon) = \max_{\tau, j} \left| \left\{ \frac{1}{\varepsilon} \int_{\tau_j}^{\tau} [G_j(\chi^{(N)}; \tau, \varepsilon) - G_j(\psi^{(N)}; \tau, \varepsilon)] d\tau \right\} \exp \frac{c_j(\tau - \tau_j)}{\varepsilon} \right| < \tag{18.144}$$

$$< \max_{\tau, j} \left\{ Amc\, |\chi_j^{(N)}(\tau, \varepsilon) - \psi_j^{(N)}(\tau, \varepsilon)| \left| \exp \frac{c_j(\tau - \tau_j)}{\varepsilon} \right| \right\} = AmcM(\varepsilon),$$

i.e.,

$$M(\varepsilon) < AmcM(\varepsilon), \tag{18.145}$$

if ε is sufficiently small. However, inequality (18.145) contradicts condition (18.96). Consequently, $\chi_j^{(N)}(\tau, \varepsilon) \equiv \psi_j^{(N)}(\tau, \varepsilon)$ $(j = 1, 2, \ldots, m)$ and the uniqueness is proven.

Thus the initially considered system (18.41) has a unique solution

$$v_j(\tau, \varepsilon) = p_j^{(N)}(\tau, \varepsilon) + \psi_j^{(N)}(\tau, \varepsilon) \exp \frac{c_j(\tau - \tau_j)}{\varepsilon} \tag{18.146}$$

$$(j = 1, 2, \ldots, m; \ N > 1)$$

in the domain $0 \leq \tau \leq L, 0 < \varepsilon \leq \varepsilon_1$. Then it follows from relationships (18.64), (18.77), (18.89) and (18.102) that the formal series (18.64) is an asymptotically convergent series in this domain.

Thus we have proved Theorem 3.4 only for $\tau \in [0, L_1]$ $(L_1 \leq L)$. This proof may be extended to the entire initial interval $[0, L]$. Indeed, in accordance with (18.69), we separate the initially assigned interval $[0, L]$ into partial intervals $[L_i, L_{i+1}]$ $(i = 0, 1, \ldots, q - 1)$ such that the end of each coincides with the beginning of the subsequent one. Then, obviously, $L_0 = 0, L_q = L$.

It is easy to see from the above that the proof of Theorem 3.4 when $0 \le \tau \le L_1$ depends substantially on inequalities (18.86), (18.87), (18.96) and on Lemma 3.3, which were subsequently used to obtain all the basic estimates. In turn, these inequalities are satisfied when the functions $\mu_j(\tau)$ $(j=1, 2, \ldots, m)$ are sufficiently well approximated by the constants c_j $(j=1, 2, \ldots, m)$.

Let us subject the partition of the interval $[0,L]$ into subintervals $[L_i, L_{i+1}]$ $(i=0, 1, \ldots, q-1)$ to two conditions: 1) the approximation of functions $\mu_j(\tau)$ by constants c_{ji} $(j=1, 2, \ldots, m;\ i=0, 1, \ldots, q-1)$ (see (18.69)) must provide for the existence of inequalities of type (18.86) and (18.87):

$$\begin{gathered} |b_{jj}^{(N)}(\tau, \varepsilon) - c_{ji}| \leqslant A, \\ |b_{jk}^{(N)}(\tau, \varepsilon)| \leqslant A \quad (j \neq k) \\ L_i \leqslant \tau \leqslant L_{i+1},\ 0 < \varepsilon \leqslant \varepsilon_1,\ c_{j0} = c_j \\ (j, k = 1, 2, \ldots, m;\ i = 0, 1, \ldots, q-1); \end{gathered} \tag{18.147}$$

2) those points in the interval $[0,L]$ at which the real part of any function $\mu_j(\tau)$ $(j=1, 2, \ldots, m)$ changes its sign (i.e., points in which the graphs of the functions Re $\mu_j(\tau)$ $(j=1, 2, \ldots, m)$ intersect the axis of the abscissas) must not fall inside the intervals $[L_i, L_{i+1}]$.

Then all the arguments presented in the hypothesis $\tau \in [0,L]$, are transferred to all subsequent subintervals. In this case, on each subinterval $[L_i, L_{i+1}]$ we seek the solution of system (18.90), which satisfies the conditions

$$\begin{gathered} \psi_j^{(N)}(\tau_{ji}\varepsilon) = \psi_j^{(N)}(\varepsilon), \\ \psi_j^{(N)}(\varepsilon) = \sum_{s=N}^{\infty} p_{js}(\tau_{ji})\varepsilon^s, \end{gathered} \tag{18.148}$$

where by analogy with (18.27)

$$\tau_{ji} = \begin{cases} L_{i+1}, & \text{if} \quad c'_{ji} \geqslant 0; \\ L_i, & \text{if} \quad c'_{ji} < 0, \end{cases} \tag{18.149}$$

where $j = 1, 2, \ldots, m;\ i = 0, 1, \ldots, q-1;$

$c'_{ji} = \operatorname{Re} c_{ji}$ (see (18.69)).

Thus, the existence and uniqueness of the solution of system (18.90), satisfying conditions of form (18.101) and (18.102), is proven for each of the subintervals $[L_i, L_{i+1}]$ (i=0, 1, ..., q-1). It is now necessary to show that the solution in each preceding subinterval transforms into a solution in the subsequent subinterval, in order to indicate the existence and uniqueness of the solution with the specified properties, over the entire interval [0,L]. /118

Indeed, the partition of interval [0,L] which provides for the approximation of functions $\mu_j(\tau)$ (j=1, 2, ..., m) by constants c_{ji} (j=1, 2, ..., m; i=0, 1, ..., q-1) under conditions that inequalities (18.147) are satisfied, is not unique. Along with the considered partition of interval [0,L] into subintervals $[L_i, L_{i+1}]$ $(i=0,1,\ldots,q-1)$, we consider a new partition into subintervals $[\bar{L}_k, \bar{L}_{k+1}]$ $(k=0,1,\ldots,q'-1;\ q'>q;\ \bar{L}_0=0,\ L_{q'}=L)$ which must have the following properties: in the first place the approximating functions $\bar{c}_j(\tau)$ determined by analogy with (18.69) provide for the fulfillment of inequalities of form (18.147), which makes it possible to carry out the proof of the existence theorem as done above for each of $[\bar{L}_k, \bar{L}_{k+1}]$ $(k=0,1,2,\ldots,q'-1)$; ; in the second place the last system of subintervals $[\bar{L}_k, \bar{L}_{k+1}]$ $(k=0, 1, \ldots, q'-1)$ must thus overlap with the first so that all L_{i+1} $(i=0,1,\ldots,p)$ lie inside some subintervals of the second system. For example, the following inequalities must exist: $0<\bar{L}_1<L_1<\bar{L}_2<L_2<\ldots<L$. We note that in this case some of the functions $\mathrm{Re}\,\mu_j(\tau)$ $(j=1,2,\ldots,m)$ may change their sign inside a particular part of the subintervals $[\bar{L}_k, \bar{L}_{k+1}]$ $(k=0,1,\ldots,q'-1;\ q'>q)$. Let us then select these subintervals of sufficiently small length so that when the function $\mu_j(\tau)$ is approximated by piece-wise constant functions $\bar{c}_j(\tau)$ of the form (18.69), the real parts of the constant approximations are equal to zero:

$$c'_{jk}=\mathrm{Re}\,c_{jk}=0.$$

On the other hand, the imaginary parts of these constants must then be different from zero, because according to the condition of the theorem, $\mu_j(\tau)$ $\neq 0$ (j=1, 2, ..., m) everywhere on [0,L]. This selection of constants which approximate the functions $\mu_j(\tau)$ (j=1, 2, ..., m) makes it possible for us, as before, to rest on Lemma 3.3. However, with τ_{jl} from (18.148) and to use substitution of variables of the form (18.89) in proving the theorem of existence and uniqueness of the solution of system (18.90) on each of the subintervals $[\bar{L}_k, L_{k+1}]$ $(k=0,1,\ldots,q'-1)$.

The theorem of existence and uniqueness for the intervals $[0,L_1]$ and $[L_1,L_2]$ is proven in the same way as for the intervals $[0,\bar{L}_1]$ and $[\bar{L}_1,\bar{L}_2]$. In this case the solution of system (18.90) coincides along the common part of the intervals $[0,L_1]$ and $[0,\bar{L}_1]$, i.e., on $[0,\bar{L}_1]$ due to the coincidences of initial conditions (18.101). Then, if the initial conditions for the solution of system (18.90) in $[\bar{L}_1,\bar{L}_2]$ are taken as value of the solution to the system at the point $\tau=\bar{L}_1$ established in the interval $[0,L_1]\supset[0,\bar{L}_1]$, then the solution of system (18.71) will be continued in the interval $[0,\bar{L}_2]$. Due to its uniqueness, this solution may be extended to the interval $[0,L_2]$, etc. Thus we can "splice" solutions established in neighboring intervals $[0,L_1]$ and $[L_1,L_2]$. Repeating the above arguments the required number of times, we find that the solution of system (18.90), which satisfies conditions

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$$\psi_j^{(N)}(0,\varepsilon)=\psi_j^{(N)}(\varepsilon) \qquad (j=1,2,\ldots,m) \tag{18.50}$$

and

$$|\psi_j^{(N)}(\tau,\varepsilon)|\leqslant K_N\varepsilon^N\exp\frac{|\xi_j(\tau)|}{\varepsilon}, \tag{18.151}$$

where

$$\xi_j(\tau)=\operatorname{Re}\mu_j(\tau) \qquad (j=1,2,\ldots,m)$$

exists and is unique in $[0,L]$.

Conditions (18.151) are obtained in the following manner. In each interval $[L_k,L_{k+1}]$, the solution of system (18.90) constructed by the method described above satisfies the inequalities

$$|\psi_j^{(N)}(\tau,\varepsilon)|\leqslant K_N\varepsilon^N\left|\exp\left(-\frac{c_{jk}(\tau-\tau_{ik})}{\varepsilon}\right)\right| \tag{18.152}$$
$$(j=1,2,\ldots,m;\quad k=0,1,\ldots,q-1;\ L_k<\tau\leqslant L_{k+1}),$$

where τ_{jk} is selected in accordance with (18.149). Then, when the partition of the interval $[0,L]$ is sufficiently fine, the right side of inequality (18.152) will be majorized by the quantity $K_N\varepsilon^N\exp\frac{|\xi_j(\tau)|}{\varepsilon}$ for any $\tau\in[0,L]$ and $j=1, 2, \ldots, m$.

Thus, by continuing the solution of system (18.90) from one subinterval to the next, as described above, we shall find a unique solution continuous in τ which satisfies the initial conditions (18.150) and conditions (18.151).

Proceeding from this "spliced" solution, which satisfies conditions (18.150) and (18.151), we now exchange two variables which are reciprocal with respect to the changes in variables which we use to transform the system of differential equations (18.41) to the form (18.90). Then inequalities (18.151) in the final analysis will give us the required asymptotic estimate for the residue of the formal series (18.64): /120

$$|p_j(\tau, \varepsilon) - p_j^{(N)}(\tau, \varepsilon)| \leqslant K_N \varepsilon^N \qquad (18.153)$$

$$(j = 1, 2, \ldots, m;\ N > 1;\ 0 \leqslant \tau \leqslant L;\ 0 < \varepsilon \leqslant \varepsilon_1).$$

Consequently series (18.64) is not simply a formal solution of the system of differential equations (18.41), but represents an asymptotic expansion in powers of the parameter ε for the solution of the above system in the domain $0 \leq \tau \leq L$, $0 < \varepsilon \leq \varepsilon_1$.

The theorem is proved.

CHAPTER 4. CONSTRUCTION OF AN ASYMPTOTIC SOLUTION WHEN THE CHARACTERISTIC EQUATION HAS MULTIPLE ROOTS

Section 19. General Remarks

The method described in the preceding chapter makes it possible to apply /121
the asymptotic method and to reduce a system of linear differential equations (12.1) or (17.1) to the corresponding systems (13.2) and (17.4), which are partitioned into isolated subsystems of a lower order compared with the initial equations. In the case when the roots of the characteristic equation (12.3) remain simple along the segment [0,L], such a partitioning, as shown in Section 16, leads to scalar differential equations which can be integrated by quadrature. Consequently, in this case the partitioning method makes it possible for us to obtain the asymptotic solution of systems (12.1) and (17.1).

In a more complex case, when multiple roots occur among the roots of the characteristic equation (12.3), the algorithm which has been presented cannot be used to obtain the asymptotic solution of the above systems because the differential equations (13.2) and (17.4) which are obtained during the partitioning process, generally speaking, cannot be integrated by quadrature (the matrices contained in the given systems have the same structure as the matrices of systems (12.1) and (17.1)). Only in the simpler case, when the multiple roots are associated with simple elementary dividers, is it possible (as shown in Section 20) to utilize asymptotic partitioning and construct the solution of initial systems while subjecting the systems resulting from partition to additional investigation. In the general case, however, when multiple roots are accompanied by multiple elementary dividers, the solution of the systems resulting from partition is rather difficult. It would be possible to use the specific structure of the systems resulting from partition and to apply to them the method discussed in Sections 21 and 23. However, we should point out that the theorems on asymptotic partitioning have been obtained under the assumption that condition (15.3) is satisfied. It is known beforehand that this condition is satisfied if among the roots of the characteristic equation there are purely imaginary roots containing multiple elementary dividers. Indeed, let us assume that the matrix $A_0(\tau)$ has the form

$$A_0(\tau) = \begin{bmatrix} i\alpha(\tau) & 1 \\ 0 & i\alpha(\tau) \end{bmatrix}, \tag{19.1}$$

where $\alpha(\tau)$ is a real positive function on the segment [0,L]. Then as pointed /122
out above, in order for condition (15.3) to be satisfied, it is necessary that the matrix

$$\Delta(\tau) = \frac{1}{2}[A_0(\tau) + A_0^*(\tau)] \tag{19.2}$$

not have eigenvalues with positive real parts. In this case,

$$\Delta(\tau) = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{19.3}$$

from which it follows that the eigenvalues of the matrix are equal to $\pm \frac{1}{2}$.

This chapter is concerned with the investigation of the case when the characteristic equation has multiple roots. It is assumed that the multiplicity of roots and the multiplicity of the corresponding elementary dividers does not vary over the segment [0,L]. In this case the method presented here makes it possible to circumvent initial partitioning and to construct the asymptotic solution for the initial systems of linear differential equations.

Thus, as before, we represent the roots of the characteristic equations (12.3) by $\lambda_1(\tau),\lambda_2(\tau), \ldots, \lambda_p(\tau)(p<n)$ and assume that on the segment [0,L] they have a constant multiplicity $k_1,k_2, \ldots, k_p$ $\left(\sum_{j=1}^{p} k_j = n\right)$.

Then we can encounter three cases depending on the behavior of their elementary dividers. Specifically, each root $\lambda_j(\tau)(j=1, 2, \ldots, p)$ may be associated with the following:

a) k_j simple elementary divider;

b) one elementary divider of multiplicity k_j; and

c) $r_j(1<r_j<k_j)$ elementary dividers of multiplicity $s_{j1}, s_{j2}, \ldots, s_{jr_j}$ $(s_{j1}+s_{j2}+\ldots+s_{jr_j}=k_j; j=1, 2, \ldots, p)$.

Below we shall consider each of the cases separately. For the sake of generality we shall investigate nonhomogeneous systems of type (17.4), isolating two cases separately as in Section 17:

1) the case of "resonance"--the function $i\nu(\tau)$ for some values $\tau\in[0,L]$ becomes equal to one of the roots of the characteristic equation (12.3), for example, to root $\lambda_1(\tau)$; however,

$$i\nu(\tau) \neq \lambda_r(\tau), \quad r = 2, 3, \ldots, p, \tag{19.4}$$

for any $\tau\in[0,L]$;

2) the case of "nonresonance," when

$$i\nu(\tau) \neq \lambda_j(\tau), \quad j = 1, 2, \ldots, p, \tag{19.5}$$

for all $\tau\in[0,L]$.

Section 20. The Case of Simple Elementary Dividers

Before constructing the solution, we recall that in this case (ref. 15) /123 the canonical matrix for matrix $A_0(\tau)$ will have a simple structure. In other

words, we can specify a nonsingular matrix $T(\tau)$ such that

$$W(\tau) = T^{-1}(\tau) A_0(\tau) T(\tau) = \begin{bmatrix} W_1(\tau) & & 0 \\ & W_2(\tau) & \\ & \ddots & \\ 0 & & W_p(\tau) \end{bmatrix}, \tag{20.1}$$

where $W_j(\tau)$ is a square matrix of order k_j which has the form

$$W_j(\tau) = \begin{bmatrix} \lambda_j(\tau) & & 0 \\ & \lambda_j(\tau) & \\ & \ddots & \\ 0 & & \lambda_j(\tau) \end{bmatrix}, \quad j = 1, 2, \ldots, p. \tag{20.2}$$

Then in this case condition (15.3) will also be satisfied when the roots are purely imaginary, because $W_0(\tau)+W^*_0(\tau)=0$. Consequently, Theorems 3.1 and 3.3 are applicable to system (17.4) in this case and make it possible to partition system (17.1) asymptotically into p isolated subsystems of the form

$$\frac{d\xi_j}{dt} = [W_j(\tau) + \varepsilon W_j(\tau, \varepsilon)]\, \xi_j^*, \qquad j = 1, 2, \ldots, p, \tag{20.3}$$

where $W_j(\tau, \varepsilon)$ are square matrices of order k_j which have the form

$$W_j(\tau, \varepsilon) = \sum_{s=0}^{m} \varepsilon^{s-1} W_{js}(\tau) \tag{20.4}$$

(m is any natural number).

Let us carry out an additional transformation of system (20.3) assuming, specifically, that

$$\xi_j(t, \varepsilon) = \exp\left\{\int_0^t \lambda_j(\tau)\, dt\right\} q_j, \qquad j = 1, 2, \ldots, p. \tag{20.5}$$

Then, taking into account (20.2), we obtain /124

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* For simplicity we consider the "nonresonant case."

$$\frac{dq_j}{dt} = \varepsilon W_j(\tau, \varepsilon) q_j, \quad j = 1, 2, \ldots, p. \tag{20.6}$$

By converting from differentiation with respect to t to differentiation with respect to $\tau = \varepsilon t$ in system (20.6) we have

$$\frac{dq_j}{d\tau} = W_j(\tau, \varepsilon) q_j, \quad j = 1, 2, \ldots, p. \tag{20.7}$$

System (20.7), unlike the initial system (17.1), makes it possible for us to apply the method of successive approximations. Consequently, by applying the method of successive approximations to system (20.7) we obtain a solution for the vector q_j in the form of a converging series in powers of the parameter ε. Then, by utilizing (20.5) we shall obtain the required solution for the vector ξ_j, j=1, ..., p.

Note. Generally speaking, the initial system of differential equations (17.1) cannot be integrated in the general case by the method of successive approximations. Indeed, let us convert from differentiation with respect to t to differentiation with respect to τ in system (17.1). We shall have

$$\frac{dx}{dt} = \frac{1}{\varepsilon} A(\tau, \varepsilon) x + \frac{1}{\varepsilon} b(\tau, \varepsilon) e^{i\theta(t,\varepsilon)}. \tag{20.8}$$

It follows that when $\varepsilon \to 0$, the matrix $\frac{1}{\varepsilon}A(\tau, \varepsilon)$ and the vector $\frac{1}{\varepsilon}b(\tau, \varepsilon)$ become infinite.

Section 21. Formal Solution When There is One Multiple Elementary Divider

Let us now consider the construction of the asymptotic solution for a system of form (17.1):

$$\frac{dx}{dt} = A(\tau, \varepsilon) x + \varepsilon b(\tau, \varepsilon) e^{i\theta(t,\varepsilon)} \tag{17.1'}$$

in the case when the root $\lambda_j(\tau)$ (j=1, 2, ..., p) is associated with an elementary divider with the same multiplicity as $\lambda_j(\tau)$. In this case, in order to obtain the asymptotic solution we shall apply the method presented in references 104-106, 107-109, 110-112 and 113-115.

We note that in this case the canonical form of matrix $A_0(\tau)$ consists of Jordan cells, i.e., it has the form (18.3).

Then we are able to prove the following theorem.

Theorem 4.1. If $A(\tau,\varepsilon), b(\tau,\varepsilon)$ and $\nu(\tau)$ on the segment $[0,L]$ contain derivatives with respect to τ of all orders, and the matrix

$$C(\tau)=T^{-1}(\tau)[A(\tau)T(\tau)-T'(\tau)] \tag{21.1}$$

is such that its elements which stand at the intersection of the r_j line and $r_{j-1}+1$ column ($r_j=k_1+k_2+\ldots+k_j$, j=1, 2, ..., p) nowhere become equal to zero, i.e.,

$$\{c(\tau)\}_{r_j,r_{j-1}+1}\neq 0 \tag{21.2}$$

for any $\tau\in[0,L]$ and all j=1, 2, ..., p, then the formal particular solution of system (17.1) in the case of "resonance" may be represented in the form

$$x(t,\varepsilon)=[U_1(\tau,\mu_1)h_1+P(\tau,\mu_1)]e^{i\theta(t,\varepsilon)}+\sum_{k=2}^{p}U_k(\tau,\mu_k)h_k, \tag{21.3}$$

where $U_j(\tau,\mu_j)$ (j=1, 2, ..., p), $P(\tau,\mu_1)$ are n-dimensional vectors and h_j are scalar functions which are determined by linear differential equations

$$\frac{dh_1}{dt}=[\lambda_1(\tau,\mu_1)-i\nu(\tau)]h_1+z(\tau,\mu_1), \tag{21.4}$$

$$\frac{dh_k}{dt}=\lambda_k(\tau,\mu_k)h_k,\quad k=2,3,\ldots,p, \tag{21.5}$$

and there exist formal expansions

$$\begin{aligned} U_j(\tau,\mu_j)&=\sum_{s=0}^{\infty}\mu_1^sU_{js}(\tau), \quad \lambda_j(\tau,\mu_j)=\sum_{s=0}^{\infty}\mu_j^s\lambda_{js}(\tau),\\ P(\tau,\mu_1)&=\sum_{s=0}^{\infty}\mu_1^sP_s(\tau), \qquad z(\tau,\mu_1)=\sum_{s=0}^{\infty}\mu_1^sz_s(\tau), \end{aligned} \tag{21.6}$$

where μ_j (j=1, 2, ..., p) is a parameter associated with the parameter ε by the following relationship

$$\mu_j=\sqrt[k_j]{\varepsilon},\qquad j=1,2,\ldots,p. \tag{21.7}$$

Proof. The proof of this theorem as well as the proof of preceding theorems, which pertain to the formal construction of a solution, consists of specifying the algorithm for determining the coefficients of formal series (21.6).

As soon as the latter are determined properly, we can find the function $h_j(t,\varepsilon)$ from the scalar equations (21.4) and (21.5) and thus utilize (21.3) to construct the required formal solution of system (17.1').

For this purpose we substitute the vector x, which is determined from relationships (21.3)-(21.5), into system (17.1'). We have /126

$$\begin{aligned}&\{\varepsilon U_1'(\tau,\mu_1)-[A(\tau,\varepsilon)-\lambda_1(\tau,\mu_1)E]U_1(\tau,\mu_1)h_1e^{i\theta(t,\varepsilon)}+\\&+\sum_{k=2}^{p}\{\varepsilon U_k'(\tau,\mu_k)-[A(\tau,\varepsilon)-\lambda_k(\tau,\mu_k)E]U_k(\tau,\mu_k)\}h_k+\\&+\{U_1(\tau,\mu_1)z(\tau,\mu_1)+\varepsilon P'(\tau,\mu_1)-\\&-[A(\tau,\varepsilon)-i\nu(\tau)E]P(\tau,\mu_1)-\varepsilon b(\tau,\varepsilon)\}e^{i\theta(t,\varepsilon)}=0.\end{aligned}\tag{21.8}$$

We now require that the coefficients in front of the functions $h_1e^{i\theta(t,\varepsilon)}$, h_k $(k=2,3,\ldots,p)$ and $e^{i\theta(t,\varepsilon)}$ are equal to zero in identity (21.8). Then we obtain

$$[A(\tau,\varepsilon)-\lambda_j(\tau,\mu_j)E]U_j(\tau,\mu_j)=\varepsilon U_j'(\tau,\mu_j)\qquad(j=1,2,\ldots,p),\tag{21.9}$$

$$[A(\tau,\varepsilon)-i\nu(\tau)E]P(\tau,\mu_1)=U_1(\tau,\mu_1)z(\tau,\mu_1)+\varepsilon[P'(\tau,\mu_1)-b(\tau,\varepsilon)\ .\tag{21.10}$$

Relationships (21.9) and (21.10) make it possible for us to determine the coefficients of the formal series (21.6).

1. First we utilize relationship (21.9), which associates the unknown vectors $U_j(\tau,\mu_j)$ and the unknown functions $\lambda_j(\tau,\mu_j)$, j=1, ..., p.

We shall now show that by proceeding from relationship (21.9), we can determine the vectors $U_j(\tau,\mu_j)$ and the functions $\lambda_j(\tau,\mu_j)$(j=1, 2, ..., p). In order not to obsure the equations obtained below with a large number of subscripts, we shall concern ourselves only with the determination of vectors $U_1(\tau,\mu_1)$ and functions $\lambda_1(\tau,\mu_1)$ at this time.

Then, by substituting j=1 into relationship (21.9) and comparing the coefficients in front of the same powers of parameter $\mu_1=\sqrt[k_1]{\varepsilon}$, we obtain

$$[A_0(\tau)-\lambda_{10}(\tau)E]U_{10}(\tau)=0,\tag{21.11}$$

$$[A_0(\tau)-\lambda_{10}(\tau)E]U_{1s}(\tau)=\sum_{k=0}^{s-1}U_{1k}(\tau)\lambda_{1s-k}(\tau)+H_{1s}(\tau)\qquad(s=1,2,\ldots),\tag{21.12}$$

$$\left[\frac{s}{k_1}\right]$$

where

$$H_{1s}(\tau) = U'_{1s-k_1}(\tau) - \sum_{k=1} A_k(\tau) U_{1,s-k_1k}(\tau)^*, \quad s = k_1, k_1+1, \ldots,$$
$$H_{1s}(\tau) \equiv 0 \qquad (s = 1, 2, \ldots, k_1 - 1), \tag{21.13}$$
$$\lambda_{10}(\tau) = \lambda_1(\tau).$$

In the future it is desirable to transform from vectors $U_{1s}(\tau)$(s=0, 1, ...) to vectors /127

$$Q_{1s}(\tau) = T^{-1}(\tau) U_{1s}(\tau). \tag{21.14}$$

Then, multiplying the vector equations (21.11) and (21.12) on the left by the matrix $T^{-1}(\tau)$ and taking into account (18.2) and (21.14), we have

$$[W_0(\tau) - \lambda_{10}(\tau) E] Q_{10}(\tau) = 0, \tag{21.15}$$

$$[W_0(\tau) - \lambda_{10}(\tau) E] Q_{1s}(\tau) = \sum_{k=0}^{s-1} Q_k(\tau) \lambda_{1s-k}(\tau) + \overline{H}_{1s}(\tau), \tag{21.16}$$

where

$$\overline{H}_{1s}(\tau) = T^{-1}(\tau) H_{1s}(\tau), \qquad s = 1, 2, \ldots . \tag{21.17}$$

Let us first consider equation (21.15). On the basis of (18.3) it breaks down into p vector equations of the form

$$[W_{r0}(\tau) - \lambda_{10}(\tau) E_{k_r}] Q_{1r0}(\tau) = 0, \quad r = 1, 2, \ldots, p, \tag{21.18}$$

where $Q_{1r0}(\tau)$ is a vector with dimension k_r(r=1, 2, ..., p) which is formed from the corresponding components of vector $Q_{10}(\tau)$.

Since

$$\det [W_{r0}(\tau) - \lambda_{10}(\tau) E_{k_r}] \neq 0, \quad r = 2, 3, \ldots, p, \tag{21.19}$$

for any $\tau \in [0, L]$, it follows from (21.18) that

$$Q_{1r0}(\tau) \equiv 0 \tag{21.20}$$

for all r=2, 3, ..., p.

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* The whole part of number r will be designated conventionally by [r].

When r=1 equation (21.18), in accordance with (18.7), transforms into equation

$$I_1 Q_{110}(\tau) = 0, \tag{21.21}$$

where I_1 has the form

$$I_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}. \tag{21.22}$$

By transforming to a coordinate form in equation (21.21) and taking into account the form of matrix I_1, we obtain

$$\{q_{10}(\tau)\}_2 = \{q_{10}(\tau)\}_3 = \dots = \{q_{10}(\tau)\}_{k_1} = 0. \tag{21.23}$$

The component $\{q_{10}(\tau)\}_1$ remains arbitrary. To obtain the nontrivial solution, we let /128

$$\{q_{10}(\tau)\} = 1. \tag{21.24}$$

Thus the vector $Q_{10}(\tau)$ has been determined. On the basis of (21.20), (21.23) and (21.24) it has the form

$$Q_{10}(\tau) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}. \tag{21.25}$$

Knowing the vector $Q_{10}(\tau)$, we determine the vector $U_{10}(\tau)$ from relationship (21.14):

$$U_{10}(\tau) = T(\tau) Q_{10}(\tau), \tag{21.26}$$

i.e., $U_{10}(\tau)$ is an eigenvector of the matrix $A_0(\tau)$ corresponding to the eigenvalue $\lambda_{10}(\tau)$.

Let us now determine the vectors $Q_{1s}(\tau)$ and the functions $\lambda_{1s}(\tau)$

(s=1, 2, ...). To do this we use the vector equation (21.16) which, according to (18.3), like equation (21.15), breaks down into p equations.

$$[W_{r0}(\tau) - \lambda_{10}(\tau) E_{r_k}] Q_{1rs}(\tau) = \sum_{k=0}^{s-1} Q_{1kr}(\tau) \lambda_{1s-k}(\tau) + \bar{H}_{1rs}(\tau) \qquad (21.27)$$
$$(r = 1, 2, \ldots, p, \quad s = 1, 2, \ldots),$$

in which $Q_{1rs}(\tau)\bar{H}_{1rs}(\tau)$ are vectors with dimension k_r(r=1, 2, ..., p), formed from the components of vectors $Q_{1s}(\tau)\bar{H}_{1s}(\tau)$.

Since for all r=2, ..., p the determinant of system (21.27) is not equal to zero, we have

$$Q_{1rs}(\tau) = [W_{r0}(\tau) - \lambda_{10}(\tau) E_{k_r}]^{-1} \Big[\sum_{k=0}^{s-1} Q_{1k_r}(\tau) \lambda_{1s-k}(\tau) + \bar{H}_{1rs}(\tau)\Big] \qquad (21.28)$$
$$(r = 2, 3, \ldots, p; \ s = 1, 2, \ldots),$$

and according to (21.13),

$$Q_{1rs}(\tau) = 0, \quad r = 2, \ldots, p; \quad s = 1, 2, \ldots, k_1 - 1. \qquad (21.29)$$

All the subsequent vectors $Q_{1rs}(\tau)$(r=2, 3, ..., p;s=k_1,k_1+1, ...) are determined from relationship (21.28). However, the right side of this relationship contains the functions $\lambda_{11}(\tau), \lambda_{12}(\tau)$, ..., which must be determined. /129

When r=1, equation (21.27), on the basis of (18.4), may be written in the following manner:

$$I_1 Q_{11s}(\tau) = \sum_{k=0}^{s-1} Q_{11k}(\tau) \lambda_{1s-k}(\tau) + \bar{H}_{11s}(\tau) \qquad (s = 1, 2, \ldots). \qquad (21.30)$$

By using this equation and taking into account the form of matrix I_1, we conclude that the first components of vectors $Q_{11s}(\tau)$(s=1, 2, ...) remain arbitrary. Therefore we may assume that they are equal to zero, i.e.,

$$q_{1s}(\tau)\}_1 = 0 \qquad (s = 1, 2, \ldots). \qquad (21.31)$$

The remaining components of vectors $Q_{11s}(\tau)$(s=1, 2, ...) as well as the components of vectors $Q_{1rs}(\tau)$(r=2, 3, ..., p;s=1, 2, ...), as follows from

equation (21.30), are expressed again in terms of the unknown functions $\lambda_{1s}(\tau)$ (s=1, 2, ...). We shall now proceed to determine these unknown functions.

Since the last line of matrix I_1 consists of zeros, the last scalar equation in system (21.30) has the form

$$\sum_{k=0}^{s-1} \{q_{1k}(\tau)\}_{k_1} \lambda_{1s-k}(\tau) + \{h_{1s}(\tau)\}_{k_1} = 0 \qquad (s = 1, 2, \ldots). \tag{21.32}$$

Equation (21.32) which has been obtained makes it possible to determine the unknown functions $\lambda_{11}(\tau), \lambda_{12}(\tau)$, ... Indeed, as we shall show below, the k_1 components of vectors $Q_{1s}(\tau)$(s=1, 2, ...) are expressed in a definite manner by means of functions $\lambda_{1s}(\tau)$(s=1, 2, ...). Therefore, if the expressions established for $\left\{q_{1k}(\tau)\right\}_{k_1}$, are substituted into equation (21.32) we can find the functions $\lambda_{1s}(\tau)$(s=1, 2, ...) from this equation.

We shall now establish the relationship mentioned above. Substituting s=1 into equation (21.30), we have

$$I_1 Q_{111}(\tau) = Q_{110}(\tau)\lambda_{11}(\tau). \tag{21.33}$$

Transforming to a coordinate form and noting the form of matrix I_1 and of the vector $Q_{110}(\tau)$ as well as (21.13), we find

$$\begin{aligned} \{q_{11}(\tau)\}_2 &= \lambda_{11}(\tau), \\ \{q_{11}(\tau)\}_3 &= \ldots = \{q_{11}(\tau)\}_{k_1} = 0. \end{aligned} \tag{21.34}$$

Let us assume that s=2. Then equation (21.30) will be written in the form

$$I_1 Q_{112}(\tau) = Q_{110}(\tau)\lambda_{12}(\tau) + Q_{111}(\tau)\lambda_{11}(\tau). \tag{21.35}$$

whence

$$\begin{aligned} \{q_{12}(\tau)\}_2 &= \lambda_{12}(\tau), \\ \{q_{12}(\tau)\}_3 &= \{q_{11}(\tau)\}_2\lambda_{11}(\tau) = [\lambda_{11}(\tau)]^2, \\ \{q_{12}(\tau)\}_4 &= \ldots = \{q_{12}(\tau)\}_{k_1} = 0. \end{aligned} \tag{21.36}$$

When s=3, we obtain the following from equation (21.30):

$$\begin{aligned} \{q_{13}(\tau)\}_2 &= \lambda_{13}(\tau), \\ \{q_{13}(\tau)\}_3 &= 2\lambda_{11}(\tau)\lambda_{12}(\tau), \\ \{q_{14}(\tau)\}_4 &= [\lambda_{11}(\tau)]^3, \\ \{q_{13}(\tau)\}_5 &= \ldots = \{q_{13}(\tau)\}_{k_1} = 0. \end{aligned} \tag{21.37}$$

Thus we can see that k_1-th components of vectors $Q_{110}(\tau), Q_{111}(\tau), Q_{112}(\tau)$ and $Q_{113}(\tau)$ are equal to zero. Assuming that s=4, ..., k_1-2 in equation (21.30), it is easy to show that the k_1-th components of vectors $Q_{114}(\tau)$, ..., $Q_{11k_1-2}(\tau)$ are also equal to zero, i.e.,

$$\{q_{1s}(\tau)\}_{k_1} = 0, \qquad 0 \leqslant s \leqslant k_1 - 2. \tag{21.38}$$

In regard to the k_1-th component of vector Q_{11k-1}, it is equal to

$$\{q_{1k_1-1}(\tau)\}_{k_1} = [\lambda_{11}(\tau)]^{k_1-1}. \tag{21.39}$$

Then by letting s=k_1 in the scalar equation (21.32) (when 1≤s≤k_1-1 this equation transforms into an identity in view of (21.13) and (21.38)), we have

$$\{q_{1k_1-1}(\tau)\}_{k_1}\lambda_{11}(\tau) + \{\bar{h}_{1k_1}(\tau)\}_{k_1} = 0. \tag{21.40}$$

From this equation and from equality (21.39), we obtain

$$\lambda_{11}(\tau) = \sqrt[k_1]{-\{h_{1k_1}(\tau)\}_{k_1}}\,. \tag{21.41}$$

Since according to (21.13) and (21.17)

$$\begin{aligned} -\{h_{1k_1}(\tau)\}_{k_1} = \{T^{-1}(\tau)[A_1(\tau)U_0(\tau) - U_0'(\tau)]\}_{k_1} = \{C(\tau)Q_0(\tau)\}_{k_1} = \\ = \{c(\tau)\}_{k_1,1}, \end{aligned} \tag{21.42}$$

we finally have

$$\lambda_{11}(\tau) = \sqrt[k_1]{\{c(\tau)\}_{k_1,1}}. \tag{21.43}$$

Thus the function $\lambda_{11}(\tau)$ has been determined. Knowing this function and using relationships (21.31) and (21.34), we find the vector

$$Q_{11}(\tau)=\begin{bmatrix}0\\ \lambda_{11}(\tau)\\ 0\\ \cdot\\ \cdot\\ \cdot\\ 0\end{bmatrix}.$$

To determine the function $\lambda_{12}(\tau)$ we let $s=k_1+1$ in equation (21.32). We have /131

$$\{q_{1k_1-1}(\tau)\}_{k_1}\lambda_{12}(\tau)+\{q_{1k_1}(\tau)\}_{k_1}\lambda_{11}(\tau)+\{h_{1k_1+1}(\tau)\}_{k_1}=0. \tag{21.44}$$

In view of (21.13), the vector $\bar{H}_{1k_1+1}(\tau)$ contains the vector $Q_{11}(\tau)$, which we have already found. Therefore, the component $\left\{\bar{h}_{1k_1+1}(\tau)\right\}_{k_1}$ in equation (21.44) is known.

In regard to component $\left\{q_{1k_1}(\tau)\right\}_{k_1}$, we use equation (21.30) to find

$$\{q_{1k_1}(\tau)\}_{k_1}=\{q_{1k_1-1}(\tau)\}_{k_1-1}\lambda_{11}(\tau)+\{q_{1k_1-2}(\tau)\}_{k_1-1}\lambda_{12}(\tau)+\{\bar{h}_{1k_1}(\tau)\}_{k_1-1}. \tag{21.45}$$

Because $\left\{q_{12}(\tau)\right\}_2=\lambda_{12}(\tau)$, we can use the recurrence formula

$$\{q_{1k_1-1}(\tau)\}_{k_1-1}=\{q_{1k_1-2}(\tau)\}_{k_1-2}\lambda_{11}(\tau)+\{q_{1k_1-3}(\tau)\}_{k_1-2}\lambda_{12}(\tau) \tag{21.46}$$

and noting that

$$\{q_{1k_1-3}(\tau)\}_{k_1-2}=[\lambda_{11}(\tau)]^{k_1-3}, \tag{21.47}$$

we can obtain the following expression from equality (21.45):

$$\{q_{1k_1}(\tau)\}_{k_1}=(k_1-1)[\lambda_{11}(\tau)]^{k_1-2}\lambda_{12}(\tau)+\{\bar{h}_{1k_1}(\tau)\}_{k_1-1}. \tag{21.48}$$

Then by substituting the value $\left\{q_{1k_1}(\tau)\right\}_{k_1}$ into equation (21.44) and solving the latter with respect to $\lambda_{12}(\tau)$, we obtain

$$\lambda_{12}(\tau)=-\frac{\{\bar{h}_{1k_1+1}(\tau)\}_{k_1}+\lambda_{11}(\tau)\{\bar{h}_{1k_1}(\tau)\}_{k_1-1}}{k_1[\lambda_{11}(\tau)]^{k_1-1}}. \tag{21.49}$$

We should note that the numerator in the right side of the last expression, in accordance with the condition of the theorem and of (21.43), is not equal to zero for any $\tau\in[0,L]$.

Thus, having determined the function $\lambda_{12}(\tau)$, we have also determined the vector $Q_{112}(\tau)$, because

$$Q_{112}(\tau)=\begin{bmatrix}0\\ \lambda_{12}(\tau)\\ |\lambda_{11}(\tau)|^2\\ 0\\ \cdot\\ \cdot\\ \cdot\\ 0\end{bmatrix}. \tag{21.50}$$

The same approach can be used to determine all the following functions $\lambda_{13}(\tau),\lambda_{14}(\tau)$, ..., and consequently the vectors $Q_{113}(\tau),Q_{114}(\tau)$, ... To do this we must substitute $s=k_1+2,k_1+3$, ... in equation (21.32) and use the initial vector equation (21.30) to find the variation in the k_1-th components of vectors $Q_{11k_1+1}(\tau),Q_{11k_1+2}(\tau)$, ... as a function of the scalar function $\lambda_{11}(\tau)$, $\lambda_{12}(\tau)$, ... /132

However, this approach to determine the unknown functions $\lambda_{11}(\tau),\lambda_{12}(\tau)$, ... is somewhat cumbersome since it requires that each time we determine the variation in the corresponding components as a function of the above functions. The establishment of this relationship, as the number s increases, frequently leads to tedious investigations.

Therefore the question arises as to whether or not we can proceed from equation (21.30) and obtain a new scalar equation which would be equivalent to equation (21.32) and which would contain only the unknown functions $\lambda_{11}(\tau)$ $\lambda_{12}(\tau)$, ...

It turns out that such an equation can be obtained if we utilize the nilpotent property of the matrix I_1, which consists of the fact that

$$I_1^m=0,\qquad m\geqslant k_1. \tag{21.51}$$

Indeed

$$I_1^2=\begin{bmatrix}0&0&1&0&\cdots&0\\ 0&0&0&1&\cdots&0\\ \cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\ 0&0&0&0&&1\\ 0&0&0&0&&0\\ 0&0&0&0&&0\end{bmatrix},\ldots,$$

$$I_1^j = \begin{bmatrix} 0 & 0 & \dots & 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & \dots & \dots & 0 \end{bmatrix} \Big\} k_1 - j, \tag{21.52}$$

which shows the validity of relationship (21.51).

It is easy to see that matrix I_1, in addition to property (21.51), also has another very important property, specifically: when the left side of vector equation (21.30) is multiplied by the matrix I_1, all its scalar equations are moved up by one place. In particular, the last equation in system (21.30), i.e., equation (21.32) in this case becomes second to last. Therefore, if we multiply /133 the left side of equation (21.30) k_1-1 times by the matrix I_1 we obtain a new vector equation

$$I_1^{k_1} Q_{11s}(\tau) = I_1^{k_1-1} \sum_{i=0}^{s-1} Q_{11i}(\tau)\lambda_{1s-i}(\tau) + I_1^{k_1-1} H_{11s}(\tau) \tag{21.53}$$
$$(s = 1, 2, \dots),$$

in which equation (21.32) will occupy the first place.

According to (21.51), equation (21.53) assumes the form

$$I_1^{k_1-1} \sum_{i=0}^{s-1} Q_{11i}(\tau)\lambda_{1s-i}(\tau) + I_1^{k_1-1}\overline{H}_{11s}(\tau) = 0. \tag{21.54}$$

We transform the left side of the last equation slightly, by using the initial system of equations (21.30). To do this we write equation (21.54) in the following manner

$$I_1^{k_1-2} \sum_{i=0}^{s-1} I_1 Q_{11i}(\tau)\lambda_{1s-i}(\tau) + I_1^{k_1-1}\overline{H}_{11s}(\tau) = 0. \tag{21.55}$$

Since the vector $I_1 Q_{110}(\tau)$, as a result of (21.21), is equal to the zero vector, while the vector $I_1 Q_{11j}(\tau)$ (j=1, 2, ...), in view of (21.30), is equal to

$$I_1 Q_{11j}(\tau) = \sum_{j_0=0}^{j} Q_{11j_0}(\tau)\lambda_{1j-j_0}(\tau) + \overline{H}_{11j}(\tau), \quad j = 1, 2, \dots, \tag{21.56}$$

equation (21.55) may be written in the form

$$I_1^{k-2}\sum_{i=1}^{s-1}\sum_{i_0=0}^{i-1}[Q_{11i_0}(\tau)\lambda_{1i-i_0}(\tau)+\overline{H}_{11i}(\tau)]\lambda_{1s-i}(\tau)+I^{k_1-1}\overline{H}_{11s}(\tau)=0,$$

or

$$I_1^{k_1-2}\sum_{i=1}^{s-1}\sum_{i_0=0}^{i-1}Q_{11i_0}(\tau)\lambda_{1i-i_0}(\tau)\lambda_{1s-i}(\tau)+I_1^{k_1-2}\sum_{i=k_1}^{s-1}\overline{H}_{11i}(\tau)\lambda_{1s-i}(\tau)+ \\ +I_1^{k_1-1}\overline{H}_{11s}(\tau)=0 \\ (s=1,2,\ldots). \tag{21.57}$$

If $k_1>2$, by applying to equation (21.57) the same transformations which were applied to equation (21.54), we obtain

$$I_1^{k_1-3}\sum_{i=2}^{s-1}\sum_{i_1=1}^{i-1}\sum_{i_0=0}^{i_1-1}Q_{11i_0}(\tau)\lambda_{1i_1-i_0}(\tau)\lambda_{1i-i_1}(\tau)\lambda_{1s-i}(\tau)+ \\ +I_1^{k_1-3}\sum_{i=k+1}^{s-1}\sum_{i_1=k}^{i-1}\overline{H}_{11i_1}\lambda_{1i-i_1}(\tau)\lambda_{1s-i}(\tau)+I_1^{k_1-2}\sum_{i=k}^{s-1}\overline{H}_{11i}\lambda_{1s-i}(\tau)+ \\ +I_1^{k_1-1}\overline{H}_{11s}(\tau)=0. \tag{21.58}$$

Repeating this process k-3 more times, we finally obtain /134

$$\sum_{i_{k_1-1}=k_1-1}^{s-1}\sum_{i_{k_1-2}=k_1-2}^{i_{k_1-1}-1}\cdots\sum_{i_0=0}^{i_1-1}Q_{11i_0}(\tau)\lambda_{1i_1-i_0}(\tau)\lambda_{1i_2-i_1}(\tau)\ldots\lambda_{1s-i_{k_1-1}}(\tau)+ \\ +F_{11s}(\tau)=0 \qquad (s=1,2,\ldots), \tag{21.59}$$

where

$$F_{11s}(\tau)=I_1^{k_1-1}\overline{H}_{11s}(\tau)+I_1^{k_1-2}\sum_{i_1=k_1}^{s-1}\overline{H}_{11i_1}\lambda_{1s-i_1}(\tau)+ \\ +I_1^{k_1-3}\sum_{i_2=k_1+1}^{s-1}\sum_{i_1=k_1}^{i_2-1}\overline{H}_{11i_1}(\tau)\lambda_{1i_2-i_1}(\tau)\lambda_{1s-i_2}(\tau)+\ldots+ \\ +\sum_{i_{k_1-1}=2k_1-2}^{s-1}\sum_{i_{k_1-2}=2k_1-3}^{i_{k_1-1}-1}\cdots\sum_{i_1=k_1}^{i_2-1}\overline{H}_{11i_1}(\tau)\lambda_{1i_2-i_1}(\tau)\ldots\lambda_{1s-i_{k_1-1}}(\tau) \\ (s=1,2,\ldots). \tag{21.60}$$

The resulting equation (21.59) is an equation which can be easily used to determine the unknown functions $\lambda_{11}(\tau), \lambda_{12}(\tau), \ldots$. We should note that since equation (21.59) was obtained from equation (21.30) by multiplying the latter by a singular matrix $I_1^{k_1-1}$, we cannot consider equation (21.59) to be equivalent to equation (21.30). Therefore, in the future we shall consider only the first scalar equation in the vector equation (21.59). According to preceding reasoning, the scalar equation is identical to equation (21.32).

Thus we let s=k in equation (21.59) (it is easy to see that when $1 \leq s \leq k_1-1$ it becomes an identity). We shall have

$$Q_{110}(\tau)[\lambda_{11}(\tau)]^{k_1} + F_{11k}(\tau) = 0. \tag{21.61}$$

From this equation, by taking into account (21.25) we find

$$\lambda_{11}(\tau) = \sqrt[k_1]{-\{f_{1k_1}(\tau)\}}. \tag{21.62}$$

Since, according to (21.60), /135

$$F_{11k_1}(\tau) = I_1^{k_1-1}\overline{H}_{11k_1}(\tau) = \begin{bmatrix} \{h_{1k_1}(\tau)\}_{k_1} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \tag{21.63}$$

we take into account (21.42) and obtain

$$\lambda_{11}(\tau) = \sqrt[k_1]{\{c(\tau)\}_{k_1 1}}. \tag{21.64}$$

Thus the function $\lambda_{11}(\tau)$, which is determined from equation (21.59), coincides with the function $\lambda_{11}(\tau)$, which is determined by equation (21.32).

Now if we let $s=k_1+1$ in equation (21.59), we have

$$k_1[\lambda_{11}(\tau)]^{k_1-1}\lambda_{12}Q_{110}(\tau) + [\lambda_{11}(\tau)]^{k_1}Q_{111}(\tau) + F_{11k_1+1}(\tau) = 0. \tag{21.65}$$

If we now take the first scalar equation, we find

$$\lambda_{12}(\tau) = -\frac{\{f_{1k_1+1}(\tau)\}_1}{k_1[\lambda_{11}(\tau)]^{k_1-1}}. \tag{21.66}$$

The component $\{f_{1k_1+1}(\tau)\}_1$ represents a function which is already known, because the vector $F_{11k_1+1}(\tau)$ is equal to

$$F_{11k_1+1}(\tau) = I_1^{k_1-1}\overline{H}_{11k_1+1}(\tau) + I_1^{k_1-2}\overline{H}_{11k_1}(\tau)\lambda_{11}(\tau). \tag{21.67}$$

Thus we have found the function $\lambda_{12}(\tau)$. Knowing this function, we can find the vector $Q_{112}(\tau)$ from (21.30) with no difficulty.

The same method can be used to determine all the subsequent functions $\lambda_{13}(\tau)$, $\lambda_{14}(\tau)$, ... In particular, if we let $s=k_1+m-1$ (m is any natural number) in equation (21.59) and assume that the functions $\lambda_{11}(\tau)$, ..., $\lambda_{1m-1}(\tau)$ are already known, we obtain

$$\lambda_{1m}(\tau) = -\frac{\{\tilde{f}_{1k_1+m-1}(\tau)\}_1}{k_1[\lambda_{11}(\tau)]^{k_1-1}}, \tag{21.68}$$

where $\{\tilde{f}_{1k_1+m-1}(\tau)\}_1$ is the first vector component

$$\begin{aligned}\tilde{F}_{11k_1+m-1}(\tau) = {} & I_1^{k_1-1}\overline{H}_{11k_1+m-1}(\tau) + I_1^{k_1-2}\sum_{j_1=k_1}^{k_1+m-2}\overline{H}_{11j_1}(\tau)\lambda_{1m+k_1-1-j_1}(\tau) + \\ & + I_1^{k_1-3}\sum_{j_2=k_1+1}^{k_1+m-2}\sum_{j_1=k_1}^{j_2-1}\overline{H}_{11j_1}\lambda_{1j_2-j_1}(\tau)\lambda_{1k_1+m-1-j_2}(\tau) + \ldots + \\ & + \sum_{j_{k_1-1}=2k_1-2}^{k_1+m-2}\sum_{j_{k_1-2}=2k_1-3}^{j_{k_1-1}-1}\cdots\sum_{j_1=k_1}^{j_2-1}\overline{H}_{11j_1}(\tau)\lambda_{1j_2-j_1}(\tau)\ldots\lambda_{1k_1+m-1-j_{k_1-1}}(\tau) + \\ & + Q_{110}(\tau)\sum_{h=1}^{k_1-1}\sum_{j_{k_1-h}=k_1-h+1}^{k_1+m-2-h}\sum_{j_{k_1-h-1}=k_1-h-1}^{j_{k_1-h}-1}\cdots\sum_{j_1=1}^{j_2-1}\lambda_{1j_1}(\tau)\lambda_{1j_2-j_1}(\tau)\ldots \\ & \ldots\lambda_{1k_1+m-h-j_{k_1-h}}(\tau)(\lambda_{11}(\tau))^{h-1}.\end{aligned} \tag{21.69}$$

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As we have recalled above, if we know the functions $\lambda_{1s}(\tau)$, s=1, 2, ...,

we can use the system of equations (21.30) to determine, quite easily, the unknown vectors $Q_{11s}(\tau)$ s=1, 2, ... In particular, by using systems of equations (21.33) and (21.35), which are obtained respectively from system (21.30) when s=1, 2, we find

$$Q_{111}(\tau)=\begin{bmatrix}0\\ \lambda_{11}(\tau)\\ 0\\ \cdot\\ \cdot\\ \cdot\\ 0\end{bmatrix},\quad Q_{112}(\tau)=\begin{bmatrix}0\\ \lambda_{12}(\tau)\\ \lambda_{11}^2(\tau)\\ 0\\ \cdot\\ \cdot\\ \cdot\\ 0\end{bmatrix}. \tag{21.70}$$

Having determined $\gamma_{1s}(\tau)$, we can proceed to find the following vectors from relationship (21.14):

$$U_{1s}(\tau)=T(\tau)Q_{1s}(\tau)\qquad (s=0,1,\ldots).$$

The method proposed here for determining the vectors $U_{1s}(\tau)$ and the functions $\lambda_{1s+1}(\tau)$ (s=0, 1, ...) may also be used to determine the vectors $U_{js}(\tau)$ and the functions $\lambda_{js+1}(\tau)$ (j=2, 3, ..., p), which are contained in the solution. The vectors

$$Q_{js}(\tau)=T^{-1}(\tau)U_{js}(\tau)\qquad (j=2,3,\ldots,p) \tag{21.71}$$

are found to be determined by means of equations which were obtained for the vectors $Q_{1s}(\tau)$ if, in the latter, we replace the function $\lambda_{1s+1}(\tau)$ by the function $\lambda_{js+1}(\tau)$

$$(j=2,3,\ldots,p;\qquad s=0,1,2,\ldots).$$

2. To prove the theorem completely, we must specify the method of determining the coefficient of vector $P(\tau,\mu_1)$ and of the function $z(\tau,\mu_1)$. For this purpose we utilize relationship (21.10). In this relationship we compare the coefficients in front of the same powers of parameter $\mu_1=\sqrt[k_1]{\varepsilon}$, and obtain

$$[A_0(\tau)-iv(\tau)E]P_s(\tau)=\sum_{j_0=0}^{s-1}U_{1j_0}(\tau)z_{s-j_0}(\tau)+\Pi_s(\tau) \tag{21.72}$$
$$(s=0,1,2,\ldots),$$

where

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$$\Pi_s(\tau) = P'_{s-k_1}(\tau) - b_{s-k_1}(\tau) - \sum_{i=1}^{\left[\frac{s}{k_1}\right]} A_i(\tau) P_{s-k_1-i}(\tau). \tag{21.73}$$

If we let

$$R_s(\tau) = T^{-1}(\tau) P_s(\tau), \tag{21.74}$$

we can use (18.2) and (21.4) to represent equation (21.72) in the following manner:

$$[W_0(\tau) - i\nu(\tau) E] R_s(\tau) = \sum_{j_0=0}^{s-1} Q_{1j_0}(\tau) z_{s-j_0}(\tau) + \overline{\Pi}_s(\tau), \tag{21.75}$$

where

$$\overline{\Pi}_s(\tau) = T^{-1}(\tau) \Pi_s(\tau) \qquad (s = 0, 1, \ldots). \tag{21.76}$$

In view of (18.3), system (21.75) breaks down into p subsystems of the form

$$[W_{j_0}(\tau) - i\nu(\tau) E] R_{js}(\tau) = \sum_{j_0=0}^{s-1} Q_{1j_0}(\tau) z_{s-j_0}(\tau) + \overline{\Pi}_{sj}(\tau), \tag{21.77}$$
$$(j = 1, 2, \ldots, p; \qquad s = 0, 1, \ldots).$$

Since

$$\det [W_{j0}(\tau) - i\nu(\tau) E] \neq 0 \tag{21.78}$$

for all $\tau \in [0, L]$ and j=2, 3, ..., p, it follows from subsystem (21.77) that

$$R_{js}(\tau) = [W_{j0}(\tau) - i\nu(\tau) E]^{-1} \Big[\sum_{j_0=0}^{s-1} Q_{1j_0}(\tau) z_{s-j_0}(\tau) + \overline{\Pi}_{sj}(\tau) \Big] \tag{21.79}$$
$$(j = 2, 3, \ldots, p; \qquad s = 0, 1, 2, \ldots).$$

For this reason we need consider only the equation

$$[W_{10}(\tau) - i\nu(\tau) E] R_{1s}(\tau) = \sum_{j_0=0}^{s-1} Q_{1j_0}(\tau) z_{s-j_0}(\tau) + \overline{\Pi}_{1s}(\tau) \tag{21.80}$$
$$(s = 0, 1, 2, \ldots).$$

In the case of "resonance," the function $i\nu(\tau)$ becomes equal to the function $\lambda_{10}(\tau)$ at some points of the segment $[0,L]$. Consequently, for these values of τ

$$\dot{W}_{10}(\tau) - i\nu(\tau)E = I_1. \tag{21.81}$$

Then equation (21.80) will be written in the form

$$I_1 R_{1s}(\tau) = \sum_{j_0=0}^{s-1} Q_{11j_0}(\tau) z_{s-j_0}(\tau) + \overline{\Pi}_{1s}(\tau). \tag{21.82}$$

In the resulting equation (21.82), the first components of the vectors $R_{1s}(\tau)$ $(s=0, 1, 2, \ldots)$ remain arbitrary. We set them equal to zero: /138

$$\{r_{1s}(\tau)\}_1 = 0 \qquad (s = 0, 1, 2, \ldots). \tag{21.83}$$

To determine the remaining components of the vector $R_{1s}(\tau)$ we proceed to equation (21.82) in coordinate form. We shall have

$$\{r_{1s}(\tau)\}_l = \sum_{j_0=0}^{s-1} \{q_{1j_0}(\tau)\}_{l-1} z_{s-j_0}(\tau) + \{p_{1s}(\tau)\}_{l-1}, \tag{21.84}$$

$$(2 \leqslant l \leqslant k_1),$$

$$\sum_{j_0=0}^{s-1} \{q_{1j_0}(\tau)\}_{k_1} z_{s-j_0}(\tau) + \{p_{1s}(\tau)\}_{k_1} = 0 \tag{21.85}$$

$$(s = 0, 1, 2, \ldots).$$

The resulting equation (21.85) makes it possible to determine the unknown functions $z_s(\tau)$, $s=0, 1, \ldots$, and if we know the last of equations (21.84) we can find the components of the vector $R_{1s}(\tau)$. Indeed, according to (21.38), equation (21.85) may be written in the form

$$\sum_{j_0=k_1-1}^{s-1} \{q_{1j_0}(\tau)\}_{k_1} z_{s-j_0}(\tau) + \{p_{1s}(\tau)\}_{k_1} = 0. \tag{21.86}$$

In view of (21.73) and (21.76), the last equation becomes an identity for all $0 \leq s \leq k_1 - 2$. When $s = k_1 - 1$, we obtain

$$[\lambda_{11}(\tau)]^{k_1-1} z_0(\tau) = 0. \tag{21.87}$$

Since from the nature of its construction, $\lambda_{11}(\tau)$ does not become equal to zero anywhere on the segment [0,L], it follows from (21.87) that

$$z_0(\tau) = 0. \tag{21.88}$$

Now by letting $s=k_1$ in equation (21.86), we have

$$[\lambda_{11}(\tau)]^{k_1-1} z_1(\tau) + \{p_{1k_1}(\tau)\}_{k_1} = 0. \tag{21.89}$$

It follows from this that

$$z_1(\tau) = -\frac{\{p_{1k_1}(\tau)\}_{k_1}}{[\lambda_{11}(\tau)]^{k_1-1}}. \tag{21.90}$$

Assuming that the functions $z_0(\tau), z_1(\tau), \ldots, z_{m-1}(\tau)$, where m is any natural number, are already known, we can proceed from equation (21.86) and also find the function $z_m(\tau)$. Indeed, by letting $s=k_1+m-1$ in equation (21.86), we obtain /139

$$[\lambda_{11}(\tau)]^{k_1-1} z_m(\tau) + \sum_{j_0=k_1}^{k_1+m-2} \{q_{1j_0}(\tau)\}_{k_1} z_{s-j_0}(\tau) + \{p_{1k_1+m-1}(\tau)\}_{k_1} = 0, \tag{21.91}$$

or

$$z_m(\tau) = -\frac{\sum_{j_0=k_1}^{k_1+m-2} \{q_{1j_0}(\tau)\}_{k_1} z_{s-j_0}(\tau) + \{p_{1k_1+m-1}(\tau)\}_{k_1}}{[\lambda_{11}(\tau)]^{k_1-1}}. \tag{21.92}$$

Thus, for any natural m our algorithm makes it possible for us to determine the function $z_m(\tau)$. Knowing the latter, we can use relationship (21.84) to determine the components of vectors $R_{1s}(\tau)$ (s=0, 1, ...). Indeed, by substituting s=0 in this function, we obtain

$$\{r_{10}(\tau)\}_l = \{q_{10}(\tau)\}_{l-1} z_0(\tau) = 0. \tag{21.93}$$

From this equation and from (21.88) we find

$$\{r_{10}(\tau)\}_l = 0, \quad 2 \leqslant l \leqslant k_1. \tag{21.94}$$

When s=1, we obtain the following from (21.84):

$$\{r_{11}(\tau)\}_l = \{q_{10}(\tau)\}_{l-1} z_1(\tau), \tag{21.95}$$

or, taking into account the form of vector $Q_{10}(\tau)$, we obtain

$$\{\dot{r}_{11}(\tau)\}_2 = z_1(\tau),$$
$$\{r_{11}(\tau)\}_3 = \ldots = \{r_{11}(\tau)\}_{k_1} = 0. \tag{21.96}$$

In a similar manner we determine all the subsequent vectors $R_{12}(\tau), R_{13}(\tau)$, ... When we know these we can use relationship (21.74) to determine the vectors

$$R_s(\tau) = T(\tau) R_s(\tau) \qquad (s = 0, 1, \ldots). \tag{21.97}$$

Thus, by specifying the method of determining the coefficients of formal series (21.6), we have proven Theorem 4.1.

Note 1. The theorem which we have just proved makes it possible to obtain not only the formal particular solution of system (17.1'), but also the formal general solution of the specified system. Indeed, the function $\lambda_{11}(\tau)$ (it can be shown that all the subsequent functions $\lambda_{1s}(\tau)$ (s=2, ...) as well) in accordance with (21.43) has k_1 different values. Consequently for the eigenvalue $\lambda_{10}(\tau)$, by proceeding from (21.4), we obtain k_1 different linearly independent particular solutions. Then we shall obtain k_j particular formal linearly independent solutions for the eigenvalue $\lambda_{j0}(\tau)$ (j=1, ..., p). However since $\sum_{l=1}^{p} k_l = n$, we can obtain n particular linearly independent solutions in view of what has been presented above. This means that we can construct a formal general solution for system (17.1'). /140

Note 2. A theorem similar to Theorem 4.1 also exists for the case of "nonresonance."

Theorem 4.2. If the conditions of Theorem 4.1 are satisfied, the formal general solution of the system in the case of "nonresonance" may be represented in the form

$$x(t, \varepsilon) = \sum_{j=1}^{p} U_j(\tau, \mu_j) \tilde{h}_j(t, \varepsilon) + \tilde{P}(\tau, \varepsilon) e^{i\theta(t,\varepsilon)}, \tag{21.98}$$

$$\frac{d\tilde{h}_j}{dt} = \lambda_j(\tau, \mu_j) \tilde{h}_j, \qquad j = 1, 2, \ldots, p, \tag{21.99}$$

where $U_j(\tau, \mu_j), \lambda_j(\tau, \mu_j)$ are the same as in Theorem 4.1, while $\tilde{P}(\tau, \varepsilon)$ is an n-dimensional vector which can be expanded as follows:

$$\widetilde{P}(\tau, \varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s \widetilde{P}_s(\tau). \tag{21.100}$$

The proof of this theorem is reduced to the determination of the coefficients of vector $\widetilde{P}(\tau, \varepsilon)$. Repeating the same reasoning as used in the proof of Theorem 3.3, we find

$$\widetilde{P}_s(\tau) = [A_0(\tau) - iv(\tau)E]^{-1}\Phi_s(\tau), \qquad s = 0, 1, 2, \ldots, \tag{21.101}$$

where $\Phi_s(\tau)$ is determined from relationship (17.26'), in which $H_s(\tau)$ should be replaced by $\Phi_s(\tau)$.

Section 22. Example

As an example we shall consider the equation of the fourth order

$$\frac{d^4y}{dt^4} + \varepsilon p_1(\tau, \varepsilon)\frac{d^3y}{dt^3} + 2p_2(\tau, \varepsilon)\frac{d^2y}{dt^2} + \varepsilon p_3(\tau, \varepsilon)\frac{dy}{dt} + p_4(\tau, \varepsilon)\, y = \tag{22.1}$$
$$= \varepsilon b_1(\tau, \varepsilon)\, e^{i\theta(t,\varepsilon)},$$

where $p_i(\tau, \varepsilon)$ $(i=1, 2, 3, 4)$, $b_1(\tau, \varepsilon)$ are real functions which can be differentiated a sufficient number of times with respect to $\tau=\varepsilon t$ and which can be represented in the form

$$p_i(\tau, \varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s p_{is}(\tau), \quad b_1(\tau, \varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s b_{s1}(\tau), \tag{22.2}$$
$$i = 1, 2, 3, 4,$$

where ε is a small real parameter.

By making the following substitutions /141

$$y = x_1, \quad \frac{dy}{dt} = x_2, \quad \frac{d^2y}{dt^2} = x_3, \quad \frac{d^3y}{dt^3} = x_4 \tag{22.3}$$

equation (22.1) is reduced to an equivalent system of equations

$$\frac{dx_1}{dt} = x_2, \quad \frac{dx_2}{dt} = x_3, \quad \frac{dx_3}{dt} = x_4, \tag{22.4}$$
$$\frac{dx_4}{dt} = -[p_4(\tau, \varepsilon)\, x_1 + \varepsilon p_3(\tau, \varepsilon)\, x_2 + 2p_2(\tau, \varepsilon)\, x_3 + \varepsilon p_1(\tau, \varepsilon)\, x_4] +$$
$$+ \varepsilon b_1(\tau, \varepsilon)\, e^{i\theta(t,\varepsilon)}.$$

By introducing the vectors

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad b(\tau, \varepsilon) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ b_1(\tau, \varepsilon) \end{bmatrix} \tag{22.5}$$

and the matrix

$$A(\tau, \varepsilon) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -p_4(\tau, \varepsilon) & -\varepsilon p_3(\tau, \varepsilon) & -2p_2(\tau, \varepsilon) & -\varepsilon p_1(\tau, \varepsilon) \end{bmatrix} \tag{22.6}$$

we can write system (22.4) in the form (17.1').

We shall assume in the future that the following relationships exist on the segment [0,L]:

$$p_{40}(\tau) \equiv [p_{20}(\tau)]^2, \quad p_{20}(\tau) > 0. \tag{22.7}$$

As a result of this the roots of characteristic equation (12.3) will have the form

$$\begin{aligned} \lambda_{10}(\tau) \equiv \lambda_{30}(\tau) = i\sqrt{p_{20}(\tau)}, \\ \lambda_{20}(\tau) \equiv \lambda_{40}(\tau) = -i\sqrt{p_{20}(\tau)}, \end{aligned} \tag{22.8}$$

..e., the matrix $A_0(\tau)$ has identically multiple eigenvalues on the segment

0,L]. We shall assume that the corresponding elementary dividers also have he same multiplicity.

Then the transformation matrix $T(\tau)$, which reduces the matrix $A_0(\tau)$ to he Jordan form, may be taken in the form /142

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ \lambda_{10}(\tau) & 1+\lambda_{10}(\tau) & \lambda_{20}(\tau) & 1+\lambda_{20}(\tau) \\ \lambda_{10}^2(\tau) & \lambda_{10}(\tau)[2+\lambda_{10}(\tau)] & \lambda_{20}^2(\tau) & \lambda_{20}(\tau)[2+\lambda_{20}(\tau)] \\ \lambda_{10}^3(\tau) & \lambda_{10}^2(\tau)[3+\lambda_{10}(\tau)] & \lambda_{20}^3(\tau) & \lambda_{20}^2(\tau)[3+\lambda_{20}(\tau)] \end{bmatrix} \tag{22.9}$$

he matrix which is reciprocal of the matrix $T(\tau)$ has the form

$$T^{-1}(\tau)=\frac{1}{4}\begin{bmatrix} 2+\lambda_{10}(\tau) & \dfrac{\lambda_{10}(\tau)+3}{\lambda_{10}(\tau)} & -\dfrac{1}{\lambda_{10}(\tau)} & -\dfrac{1+\lambda_{10}(\tau)}{\lambda_{10}^3(\tau)} \\ -\lambda_{10}(\tau) & -1 & \dfrac{1}{\lambda_{10}(\tau)} & \dfrac{1}{\lambda_{10}^2(\tau)} \\ 2+\lambda_{20}(\tau) & \dfrac{\lambda_{20}(\tau)+3}{\lambda_{20}(\tau)} & -\dfrac{1}{\lambda_{20}(\tau)} & -\dfrac{1+\lambda_{20}(\tau)}{\lambda_{20}^3(\tau)} \\ -\lambda_{20}(\tau) & -1 & \dfrac{1}{\lambda_{20}(\tau)} & \dfrac{1}{\lambda_{20}^2(\tau)} \end{bmatrix}. \qquad (22.10)$$

In this case the canonical matrix $W_0(\tau)$ may be represented in the following manner:

$$W_0(\tau)=\begin{bmatrix} i\sqrt{p_{20}(\tau)} & 1 & 0 & 0 \\ 0 & i\sqrt{p_{20}(\tau)} & 0 & 0 \\ 0 & 0 & -i\sqrt{p_{20}(\tau)} & 1 \\ 0 & 0 & 0 & -i\sqrt{p_{20}(\tau)} \end{bmatrix}. \qquad (22.11)$$

The functions $\{c(\tau)\}_{r_{j-1}+1}$ $(j=1,2;\ r_j=k_1+k_2)$, which in this case are transformed into functions $\{c(\tau)\}_{21}$, $\{c(\tau)\}_{41}$, , have the following form respectively

$$\{c(\tau)\}_{21}=$$
$$=\frac{2p_{21}(\tau)\lambda_{10}^2(\tau)+p_{10}(\tau)\lambda_{10}^3(\tau)+2\lambda_{10}^2(\tau)\lambda_{10}'(\tau)-p_{41}(\tau)-p_{31}(\tau)\lambda_{10}(\tau)}{4\lambda_{10}^2(\tau)},$$
$$\{c(\tau)\}_{41}= \qquad (22.12)$$
$$=\frac{2p_{21}(\tau)\lambda_{20}^2(\tau)+p_{10}(\tau)\lambda_{20}^3(\tau)+2\lambda_{20}^2(\tau)\lambda_{20}'(\tau)-p_{41}(\tau)-p_{31}(\tau)\lambda_{20}(\tau)}{4\lambda_{20}^2(\tau)}.$$

Consequently, if the functions $\{c(\tau)\}_{21}$, $\{c(\tau)\}_{41}$ do not become equal to zero anywhere on the segment $[0,L]$, then in order to obtain the formal solution of the system we can use Theorems 4.1 and 4.2. In particular, according to Theorem 4.1, the formal solution of system (22.4) in the first approximation can be represented in the form /143

$$x(t,\varepsilon)=\{[U_{10}(\tau)+\sqrt{\varepsilon}U_{11}(\tau)]\,[c_1 e^{\int_0^t(\lambda_{10}(\tau)-i\nu(\tau)+\sqrt{\varepsilon}\lambda_{11}(\tau))dt}+$$
$$+\sqrt{\varepsilon}\int_0^t z_1(\tau_1)\,e^{\int_{t_1}^t(\lambda_{10}(\tau)-i\nu(\tau)+\sqrt{\varepsilon}\lambda_{11}(\tau))dt}\,dt_1]+\sqrt{\varepsilon}p_1(\tau)\}\,e^{i\theta(t,\varepsilon)}+$$

$$+ c_2 e^{\int_0^t [\lambda_{20}(\tau)+\sqrt{\varepsilon}\lambda_{21}(\tau)]dt} [U_{20}(\tau) + \sqrt{\varepsilon}U_{21}(\tau)], \tag{22.13}$$

where

$$\lambda_{11}(\tau) = +\sqrt{\{c(\tau)\}_{21}},$$
$$\lambda_{21}(\tau) = +\sqrt{\{c(\tau)\}_{41}},$$
$$U_{j0}(\tau) = \begin{bmatrix} 1 \\ \lambda_{j0}(\tau) \\ \lambda_{j0}^2(\tau) \\ \lambda_{j0}^3(\tau) \end{bmatrix}, \qquad j = 1, 2,$$
$$U_{j1}(\tau) = \begin{bmatrix} 1 \\ 1 + \lambda_{j0}(\tau) \\ \lambda_{j0}(\tau)[2 + \lambda_{j0}(\tau)] \\ \lambda_{j0}^2(\tau)[3 + \lambda_{j0}(\tau)] \end{bmatrix} \lambda_{j1}(\tau), \quad j = 1,2, \tag{22.14}$$
$$P_1(\tau) = \begin{bmatrix} 1 \\ 1 + \lambda_{10}(\tau) \\ \lambda_{10}(\tau)[2 + \lambda_{10}(\tau)] \\ \lambda_{10}^2(\tau)[3 + \lambda_{10}(\tau)] \end{bmatrix} z_1(\tau),$$

where

$$z_1(\tau) = \frac{1}{4\lambda_{10}^3(\tau)}, \tag{22.15}$$

c_1 and c_2 are constants of integration.

If we have the solution of system (22.4), it is easy to use (22.3) and to write the solution of equation (22.1) for the case of "resonance."

In the same way, by utilizing Theorem 4.2 we can also obtain the solution of equation (22.1) for the case of "nonresonance."

Section 23. The Asymptotic Nature of the Solution

In the present section we shall show that the formal solutions constructed in Section 21, under certain conditions, have an asymptotic nature in the sense that the vector $x^{(m)}(t, \varepsilon)$ formed from the vector $x(t, \varepsilon)$ by breaking off the formal series at the m-th term differs by an arbitrarily small amount in the norm from the exact solution of system (17.1') when m is fixed and when the values of the parameter ε are sufficiently small. /144

In this connection, in the future we shall consider only the case of "resonance" in detail. The case of "nonresonance" may be investigated in the same manner.

Thus we consider the vector

$$x^{(m)} = [U_1^{(m)}(\tau,\ \mu_1)\, h_1^{(m)} + P^{(m)}(\tau,\ \mu_1)]\, e^{i\theta(t,\varepsilon)} + \sum_{k=2}^{p} U_k^{(m)}(\tau,\ \mu_k)\, h_k^{(m)}, \tag{23.1}$$

$$\frac{dh_1^{(m)}}{dt} = [\lambda^{(m)}(\tau,\ \mu_1) - i\nu(\tau)]\, h_1^{(m)} + z^{(m)}(\tau,\ \mu_1), \tag{23.2}$$

$$\frac{dh_k^{(m)}}{dt} = \lambda_k^{(m)}(\tau,\ \mu_k)\, h_k^{(m)} \qquad (k = 2,\ 3,\ldots,\ p), \tag{23.3}$$

where

$$\begin{gathered} U_j^{(m)}(\tau,\ \mu_j) = \sum_{s=0}^{m} \mu_j^s U_{js}(\tau),\ \lambda_j^{(m)}(\tau,\ \mu_j) = \sum_{s=0}^{m} \mu_j^s \lambda_{js}(\tau), \\ P^{(m)}(\tau,\ \mu_1) = \sum_{s=0}^{m} \mu_1^s P_s(\tau),\ z^{(m)}(\tau,\ \mu_1) = \sum_{s=0}^{m} \mu_1^s z_s(\tau). \end{gathered} \tag{23.4}$$

Then we can prove the following lemma.

Lemma 4.1. If the conditions of Theorem 4.1 are also satisfied for $\tau \in [0, L]$

$$\operatorname{Re}\left(\sum_{s=0}^{k_j-1} \mu_j^s \lambda_{js}(\tau)\right) \leqslant 0 \tag{23.5}$$

for all j=1, 2, ..., p, then

$$\frac{dx^{(m)}}{dt} = A(\tau,\ \varepsilon)\, x^{(m)} + b(\tau,\ \varepsilon)\, e^{i\theta(t,\varepsilon)} + \mu_1^{m+2-k_1} R_m^*(\tau,\ \mu_1), \tag{23.6}$$

where $R_m^*(\tau, \mu_1)$ is a vector which is regular with respect to the parameter $\mu_1 = \sqrt[k_1]{\varepsilon}$ in the neighborhood of the point $\varepsilon=0$.

To prove the lemma, we substitute the vector $x^{(m)}$ into the following expression: /145

$$\frac{dx^{(m)}}{dt} - A(\tau, \varepsilon)\, x^{(m)} - \varepsilon b(\tau, \varepsilon)\, e^{i\theta(t,\varepsilon)}.$$

We shall have

$$\frac{dx^{(m)}}{dt} - A(\tau, \varepsilon)\, x^{(m)} - \varepsilon b(\tau, \varepsilon)\, e^{i\theta(t,\varepsilon)} =$$

$$= \left\{ \varepsilon \frac{dU_1^{(m)}(\tau, \mu_1)}{d\tau} h_1^{(m)} + U_1^{(m)}(\tau, \mu_1)\left[(\lambda_1^{(m)}(\tau, \mu_1) - i\nu(\tau))\, h_1^{(m)} + \right.\right.$$

$$+ z^{(m)}(\tau, \mu_1)\Big] + \varepsilon \frac{dP^{(m)}(\tau, \mu_1)}{d\tau} +$$

$$\left.\left. + i\nu(\tau)\left[U_1^{(m)}(\tau, \mu_1)\, h_1^{(m)} + P^{(m)}(\tau, \mu_1)\right]\right\} e^{i\theta(t,\varepsilon)} +$$

$$+ \sum_{k=2}^{p} \left[\varepsilon \frac{dU_k^{(m)}}{d\tau} h_k^{(m)} + U_k^{(m)}(\tau, \mu_k)\, \lambda_k^{(m)}(\tau, \mu_k)\, h_k^{(m)} \right] -$$

$$- A(\tau, \varepsilon)\left\{\left[U_1^{(m)}(\tau, \mu_1)\, h_1^{(m)} + P^{(m)}(\tau, \mu_1)\right] e^{i\theta(t,\varepsilon)} + \right.$$

$$\left. + \sum_{k=2}^{p} U_k^{(m)}(\tau, \mu_k)\, h_k^{(m)} \right\} - \varepsilon b(\tau, \varepsilon)\, e^{i\theta(t,\varepsilon)}$$

or

$$\frac{dx^{(m)}}{dt} = A(\tau, \varepsilon)\, x^{(m)} + \varepsilon b(\tau, \varepsilon)\, e^{i\theta(t,\varepsilon)} + \left[U_1^{(m)}(\tau, \mu_1)\, \lambda_1^{(m)}(\tau, \mu_1) + \right.$$

$$\left. + \varepsilon \frac{dU_1^{(m)}}{d\tau} - A(\tau, \varepsilon) U_1^{(m)}(\tau, \mu_1)\right] h_1^{(m)} e^{i\theta(t,\varepsilon)} +$$

$$+ \sum_{k=2}^{p} \left[U_k^{(m)}(\tau, \mu_k)\, \lambda_k^{(m)}(\tau, \mu_k) + \varepsilon \frac{dU_k^{(m)}}{d\tau} - \right.$$

$$\left. - A(\tau, \varepsilon) U_k^{(m)}(\tau, \mu_k)\right] h_k^{(m)} +$$

$$+ \left\{U_1^{(m)}(\tau, \mu_1)\, z^{(m)}(\tau, \mu_1) + \varepsilon \frac{dP^{(m)}}{d\tau} - \right.$$

$$\left. - \left[A(\tau, \varepsilon) - i\nu(\tau) E\right] P^{(m)}(\tau, \mu_1) - \varepsilon b(\tau, \varepsilon)\right\} e^{i\theta(t,\varepsilon)}. \tag{23.7}$$

The vectors in the right side of relationship (23.7) which stand in front of the functions $h_1^{(m)} e^{i\theta(t,\varepsilon)}$, $h_k^{(m)}$ (k=2, 3, ..., p) and $e^{i\theta(t,\varepsilon)}$, according to

Theorem 4.1, are infinitesimals of the respective orders μ_1^{m+1}, μ_k^{m+1} (k=2, ..., p) /146 and μ_1^{m+1}. Therefore,

$$\frac{dx^{(m)}}{dt} = A(\tau,\ \varepsilon)\,x^{(m)} + b(\tau,\ \varepsilon)\,e^{i\theta(t,\varepsilon)} + \mu_1^{m+1}R_{1m}(\tau,\ \mu_1)\,h_1^{(m)} + \\ + \sum_{k=2}^{p} \mu_k^{m+1}R_{km}(\tau,\ \mu_k)\,h_k^{(m)} + \mu_1^{m+1}F_m(\tau,\ \mu_1), \tag{23.8}$$

where the vectors $R_{jm}(\tau,\mu_j), F_m(\tau,\mu_1)$ are bounded in the neighborhood of the point μ_j=0(j=1, 2, ..., p).

Integrating equations (23.2) and (23.3) and taking into account (23.5), we find

$$|h_1^{(m)}| \leqslant \frac{N_1}{\mu_1^{k_1-1}},\ |h_k^{(m)}| \leqslant N_k,\ k = 2,\ 3,\ldots,\ p, \tag{23.9}$$

where N_j are constants which do not depend on the parameter μ_j(j=1, 2, ..., p) which varies in the interval $0<\mu_j\leq\mu_{j0}$.

Then the system of differential equations (23.8) may be written in the form

$$\frac{dx^{(m)}}{dt} = A(\tau,\ \varepsilon)\,x^{(m)} + b(\tau,\ \varepsilon)\,e^{i\theta(t,\varepsilon)} + \mu_1^{m+2-k_1}\bar{R}_{1m}(\tau,\ \mu_1) + \\ + \sum_{k=2}^{p} \mu_k^{m+1}\,\bar{R}_{km}(\tau,\ \mu_k) + \mu_1^{m+1}F_m(\tau,\ \mu_1), \tag{23.10}$$

where $R_{jm}(\tau,\mu_j)$ are vectors bounded in the neighborhood μ_j=0(j=1, 2, ..., p).

Let us number the roots of equation (12.3) such that

$$k_1 \geqslant k_2 \geqslant \ldots \geqslant k_p. \tag{23.11}$$

Then

$$\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_p. \tag{23.12}$$

Consequently

$$\mu_1^{m+2-k_1}\bar{R}_{1m} + \sum_{k=2}^{p} \mu_k^{m+1}\,\bar{R}_{km}(\tau,\ \mu_k) + \mu_1^{m+1}F_m(\tau,\ \mu_1) = O(\mu_1^{m+2-k_1}). \tag{23.13}$$

The validity of relationship (23.6) follows from this.

Theorem 4.3. When the conditions of Lemma 4.1 are satisfied and when t=0

$$x = x^{(m)}, \tag{23.14}$$

then for any L>0 and $0<\mu_j \leq \mu_{j0}$ (j=1, 2, ..., p) we can specify a constant C, which does not depend on ε such that

$$\| x - x^{(m)} \| \leq \mu_1^{m+2-2k_1} C. \tag{23.15}$$

To prove this theorem we shall represent the exact solution of system (17.1') in the form /147

$$x = [U_1^{(m)}(\tau, \mu_1) q_1 + P^{(m)}(\tau, \mu_1)] e^{i\theta(t,\varepsilon)} + \sum_{k=2}^{p} U_k^{(m)}(\tau, \mu_k) q_k. \tag{23.16}$$

Then the vector

$$y = x - x^{(m)} = \sum_{j=1}^{p} U_j^{(m)}(\tau, \mu_j) \eta_j, \tag{23.17}$$

where

$$\eta_1 = (q_1 - h_1^{(m)}) e^{i\theta(t,\varepsilon)}, \quad \eta_k = q_k - h_k^{(m)} \qquad (k = 2, 3, \ldots, p), \tag{23.18}$$

will satisfy the system

$$\frac{dy}{dt} = A(\tau, \varepsilon) y - \mu_1^{m+2-k_1} R_m^*(\tau, \mu_1). \tag{23.19}$$

By using (23.17) we can represent the resulting system (23.19) in the following manner:

$$\sum_{j=1}^{p} U_j^{(m)}(\tau, \mu_j) \left[\frac{d\eta_j}{dt} - \lambda_j^{(m)}(\tau, \mu_j) \eta_j \right] =$$

$$= \sum_{j=1}^{p} \left[A(\tau, \varepsilon) U_j^{(m)}(\tau, \mu_j) - \right.$$

$$\left. - U_j^{(m)}(\tau, \mu_j) \lambda_j^{(m)}(\tau, \mu_j) - \varepsilon \frac{dU_j^{(m)}}{d\tau}(\tau, \mu_j) \right] \eta_j -$$

$$- \mu_1^{m+2-k_1} R_m^*(\tau, \mu_1)$$

or

$$\sum_{j=1}^{p} U_j^{(m)}(\tau,\ \mu_j)\left[\frac{d\eta_j}{dt} - \lambda_j^{(m)}(\tau,\ \mu_j)\eta_j\right] = \\ = \sum_{j=1}^{p} \mu_j^{m+1} R_{jm}(\tau,\ \mu_j)\eta_j - \mu_1^{m+2-k_1} R_m^*(\tau,\ \mu_1). \tag{23.20}$$

We shall consider system (23.20) as an algebraic system with respect to /148
the scalar quantities $\frac{d\eta_j}{dt}=\lambda_j^{(m)}(\tau,\mu_j)\eta_j$ ($j=1,\ 2,\ \ldots,\ p$). Proceeding from the construction of vectors $U_j^{(m)}(\tau,\mu_j)$*, it is easy to show that when $0<\mu_j\leq\mu_{j0}$ the rank of the given system is equal to p. Therefore, the given system can be solved and

$$\frac{d\eta_j}{dt} = \lambda_j^{(m)}(\tau,\ \mu_j)\eta_j + \sum_{i=1}^{p} \mu_i^{m+1}\alpha_{ji}(\tau,\ \mu_i)\eta_i + \mu_1^{m+2-k_1}\beta_j(\tau,\ \mu_1) \\ (j = 1,\ 2,\ldots,\ p), \tag{23.21}$$

where $\alpha_{ji}(\tau,\mu_i),\beta_j(\tau,\mu_1)$ are functions bounded in the neighborhood of the point $\mu_i=0$ ($i=1,\ 2,\ \ldots,\ p$).

Then by integrating system (23.21) and noting that $\eta_i=0$ when $t=0$, we find

$$\eta_j = \mu_1^{m+2-k_1}\int_0^t \beta_j(\tau_1,\ \mu_1)\, e^{\int_{t_1}^{t}\lambda_j^{(m)}(\tau,\mu_j)dt}\, dt_1 + \\ + \int_0^t \sum_{i=1}^{p} \mu_i^{m+1}\alpha_{ji}(\tau_1,\ \mu_i)\eta_i e^{\int_{t_1}^{t}\lambda_j^{(m)}(\tau,\mu_j)dt}\, dt_1. \tag{23.22}$$

Assuming that $m\geq k_j-1$, we obtain the following estimate from relationship (23.22) in accordance with (23.5):

$$|\eta_j| \leqslant \mu_1^{m+2-k_1}\beta_j^* N_j^* t + N_j^*\int_0^t \sum_{i=1}^{p} \alpha_{ji}^*\mu_i^{m+1}|\eta_i|\, dt, \tag{23.23}$$

where

$$\beta_j^* = \max |\beta_j(\tau,\ \mu_1)|,\ \alpha_{jl}^* = \max |\alpha_{jl}(\tau,\ \mu_l)|,$$

$$N_j^* = \max e^{\int_{t_1}^{\tau} \lambda_j^{(m)}(\tau, \mu_j) dt}, \quad \tau \in [0,\ L],\ 0 < \mu_j \leqslant \mu_{j0} \qquad (i, j = 1, 2, \ldots, p). \tag{23.24}$$

By designating

$$\max_i \alpha_{jl}^* = \alpha_j^* \tag{23.25}$$

and taking into account (23.12), we obtain the following from inequality (23.23):

$$|\eta_j| \leqslant \mu_1^{m+2-k_1} \beta_j^* N_j^* t + \alpha_j^* N_j^* \mu_1^{m+1} \int_0^t \sum_{i=1}^{p} |\eta_i|\, dt \quad (j = 1,\ 2, \ldots, p). \tag{23.26}$$

We introduce the function

$$\omega = \sum_{i=1}^{p} |\eta_i|. \tag{23.27}$$

Then if we add inequalities (23.26) term by term, we find

$$\omega \leqslant \mu_1^{m+1} N_1 \int_0^t \omega dt + \mu_1^{m+2-k_1} N_2 t, \tag{23.28}$$

where

$$N_1 = \sum_{j=1}^{p} \beta_j^* N_j^*,\quad N_2 = \sum_{j=1}^{p} \alpha_j N_i^*. \tag{23.29}$$

Applying Lemma 1.1 from Chapter 1 to inequality (23.28), we have

$$\omega \leqslant \mu_1^{m+2-2k_1} C_{1m}, \tag{23.30}$$

where

$$C_{1m} = N_2 L e^{\mu_{10}^{m+1-k_1} N_1 L}. \tag{23.31}$$

Then it follows from equality (23.27) and inequality (23.30) that

$$|\eta_j| \leqslant C_{1m} \mu_1^{m+2-2k_1} \quad (j = 1,\ 2, \ldots, p). \tag{23.32}$$

---

* $U_j^{(m)}(\tau, \mu_j) = U_{j0}(\tau) + \sum_{s=1}^{m} \mu_j^s U_{js}(\tau) \to U_{j0}(\tau)$ for $\mu_j \to 0$.

If we convert to the norm in equality (23.17), we obtain

$$\| x - x^{(m)} \| \leqslant \sum_{j=1}^{p} \| U_j^{(m)}(\tau, \mu_j) \| | \eta_j |. \tag{23.33}$$

Since the vectors $U_j^{(m)}(\tau, \mu_j)$ (j=1, 2, ..., p) are differentiable with respect to τ on the segment [0,L], then for all

$$\tau \in [0, L] \text{ and } 0 < \mu_j \leqslant \mu_{j0} \ (j = 1, 2, \ldots, p), \tag{23.34}$$

$$\| U_j^{(m)}(\tau, \mu_j) \| \leqslant M,$$

where M is a constant which does not depend on ε.

Consequently relationships (23.30), (23.33) and (23.34) prove the inequality (23.15), in which /150

$$C = pMC_{1m}. \tag{23.35}$$

The theorem is proved.

We note that the resulting estimate (23.15) is what points to the asymptotic nature of the solution $x^{(m)}$. It follows from inequality (23.15) that for all $m > 2k_1 - 2$ $x^{(m)} \to x$ when $\mu_1 \to 0$.

Note. By using the method presented in the present section, we can also evaluate the case of "nonresonance" in the same way as evaluated (23.15). It is found that in this case the evaluation is improved, specifically: when the conditions of Theorem 4.3 are satisfied in the case of "nonresonance" we have the relationship

$$\| x - x^{(m)} \| \leqslant \mu_1^{m+1-k_1} C, \tag{23.36}$$

where C is a constant which is independent of ε. It follows from inequality (23.36) that for all $m > k_1 - 1$ $x_m \to x$ when $\mu_1 \to 0$.

Section 24. The Asymptotic Solution When Several Multiple Elementary Dividers are Present

In Section 21, a detailed description was given for the process for constructing a formal solution of system (17.1') in the case when the multiplicity of elementary dividers coincides with the multiplicity of corresponding roots of the characteristic equation (12.3).

In the present section we shall present the basic results which have

been obtained in reference 111 for the more general case when several multiple elementary dividers correspond to some root of the characteristic equation.

Thus, let us assume that $r_j (1<r_j<k_j)$ elementary dividers correspond to the root $\lambda_j(\tau)(j=1, 2, \ldots, p)$ such that each of the dividers has a respective multiplicity $s_{j1}, s_{j2}, \ldots, s_{jr_j} (s_{j1}+s_{j2}+\ldots+s_{jr_j}=k_j; j=1, 2, \ldots, p)$. In this case, by following the theory of linear algebra we can specify a nonsingular matrix $T(\tau)$ for the matrix $A_0(\tau)$ so that

$$W_0(\tau) = T^{-1}(\tau) A_0(\tau) T(\tau) = \begin{bmatrix} W_{k_1}(\tau) & & 0 \\ & W_{k_2}(\tau) & \\ & \cdot & \\ & \cdot & \\ & \cdot & \\ 0 & & W_{k_p}(\tau) \end{bmatrix}, \tag{24.1}$$

where

$$W_{k_j}(\tau) = \begin{bmatrix} W_{s_{j_1}}(\tau) & & & \\ & W_{s_{j_2}}(\tau) & & \\ & & \cdot & \\ & & \cdot & \\ & & \cdot & \\ & & & W_{s_{jr_j}}(\tau) \end{bmatrix}, \tag{24.2}$$

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$$W_{s_{jn_j}} = \begin{bmatrix} \lambda_{j_0}(\tau) & 1 & 0 \ldots 0 \\ 0 & \lambda_{j_0}(\tau) & 1 \; 0 \ldots 0 \\ \cdot & \cdot & \cdot \\ 0 & 0 & 0 \; 0 \ldots \lambda_{j_0}(\tau) \end{bmatrix}$$

$$(n_j = 1, 2, \ldots, r_j, \; j = 1, 2, \ldots, p).$$

Then the following theorem may be valid in the case of "resonance."

Theorem 4.4. If $A(\tau, \varepsilon), b(\tau, \varepsilon), \nu(\tau)$ have derivatives of all orders with respect to τ and for all $\tau \in [0, L]$ m_{jn_j} the components of the vector

$$C_j(\tau) = T^{-1}(\tau) [A(\tau) T_{m_{jn_j-1}+1}(\tau) - T'_{m_{jn_j-1}+1}(\tau)], \tag{24.3}$$

where $T_{mjn_{j-1}+1}(\tau)$ is the column of the matrix $T(\tau)$ with the number $m_{jn_{j-1}+1}$ ($m_{jn_j}=k_1+k_2+\ldots+k_{j-1}+s_{j1}+s_{j2}+\ldots+s_{jn_j}$; $n_j=1, 2, \ldots, r_j$; $j=1, 2, \ldots, p$)

satisfies the relationship

$$\{c_j(\tau)\}_{m_{jn_j}} \neq 0,^* \tag{24.4}$$

then the particular formal solution of system (17.1') may be represented in the form

$$x = \sum_{n_1=1}^{r_1} [U_{1n_1}(\tau,\ \mu_{1n_1})\, h_{1n_1} + P_{n_1}(\tau,\ \mu_{1n_1})]\, e^{i\theta(t,\varepsilon)} + \tag{24.5}$$

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$$+ \sum_{k=2}^{p} \sum_{n_k=1}^{r_k} U_{kn_k}(\tau,\ \mu_{kn_k})\, h_{kn_k},$$

$$\frac{dh_{1n_1}}{dt} = [\lambda_{1n_1}(\tau,\ \mu_{1n_1}) - i\nu(\tau)]\, h_{1n_1} + z n_1(\tau,\ \mu_{1n_1}) \tag{24.6}$$

$$(n_1 = 1,\ 2, \ldots, r_1),$$

$$\frac{dh_{kn_k}}{dt} = \lambda_{kn_k}(\tau,\ \mu_{kn_k})\, h_{kn_k} \qquad (n_k = 1,\ 2, \ldots, r_k, \tag{24.7}$$

$$k = 2,\ 3, \ldots, p),$$

where $U_{jn_j}(\tau,\mu_{jn_j})$, $P_{n_1}(\tau,\mu_{1n_1})$ are n-dimensional vectors; $h_{jn_j}, z_{n_1}(\tau,\mu_{1n_1})$ are

scalar functions which admit a formal expansion

$$U_{jn_j}(\tau,\ \mu_{jn_j}) = \sum_{s=0}^{\infty} \mu_{jn_j}^s U_{jn_js}(\tau),$$

$$\lambda_{jn_j}(\tau,\ \mu_{jn_j}) = \sum_{s=1}^{\infty} \mu_{jn_j}^s \lambda_{jn_js}(\tau) + \lambda_{j_0}(\tau), \tag{24.8}$$

$$P_{n_1}(\tau,\ \mu_{1n_1}) = \sum_{s=0}^{\infty} \mu_{1n_1}^s P_{n_1s}(\tau),\quad z_{n_1}(\tau,\ \mu_{1n_1}) = \sum_{s=0}^{\infty} \mu_{1n_1}^s z_{n_1s}(\tau)$$

$$(n_j = 1,\ 2, \ldots, r_j; \qquad j = 1,\ 2, \ldots, p),$$

where μ_{jnj} is a parameter associated with the parameter ϵ by the relationship

*

It is also assumed that on the segment [0,L] "conditional" equations (equations which do not contain the unknowns; ref. 76, page 83) are satisfied.

$$\mu_{jn_j} = \sqrt[s_{jn_j}]{\varepsilon}\ (n_j = 1,\ 2,\ldots,\ r_j,\ j = 1,\ 2,\ldots,\ p). \tag{24.9}$$

We shall not prove this theorem (the reader is referred to reference 111).

The following theorem exists for the case of "nonresonance."

Theorem 4.5. If the conditions of Theorem 4.4 are satisfied, the formal general solution of system (17.1') in the case of "nonresonance" may be represented in the form

$$x = \sum_{j=1}^{p} \sum_{n_j=1}^{r_j} U_{jn_j}(\tau,\ \mu_{jn_j})\tilde{h}_{jn_j} + \tilde{\Phi}(\tau,\ \varepsilon)\, e^{i\theta(t,\varepsilon)}, \tag{24.10}$$

$$\frac{d\tilde{h}_{jn_j}}{dt} = \lambda_{jn_j}(\tau,\ \mu_{in_j})\tilde{h}_{jn_j}.$$

$$(n_j = 1,\ 2,\ldots,\ r_j,\ j = 1,\ 2,\ldots,\ p), \tag{24.11}$$

where $U_{jn_j}(\tau, \mu_{jn_j})$. $\lambda_{jn_j}(\tau, \mu_{jn_j})$ are the same as those in Theorem 4.4, while /153

$\tilde{\Phi}(\tau, \varepsilon)$ is an n-dimensional vector which admits a formal expansion

$$\tilde{\Phi}(\tau,\ \varepsilon) = \sum_{s=1}^{\infty} \varepsilon^s \tilde{P}_s(\tau). \tag{24.12}$$

The asymptotic nature of solution (24.5) is proven by the next theorem.

Theorem 4.6. If the conditions of Theorem 4.4 are satisfied and

$$x|_{t=0} = x^{(m)}|_{t=0};\ \operatorname{Re}\left(\lambda_{j_0}(\tau) + \sum_{s=1}^{s_{jn_j}-1} \mu_{jn_j}^s \lambda_{jn_j s}\right) \leqslant 0, \tag{24.13}$$

where $x^{(m)}$ is a vector determined by relationships (24.5)-(24.7) in which the series (24.8) are broken off at the m-th terms, then for any $L>0$ and $0<\mu_{jn_j} \leq \mu_{jn_j}$, we can specify such constants C_{jn_j} as are independent of the parameter ε, so that

$$\|x - x^{(m)}\| \leqslant \sum_{j=1}^{p} \sum_{n_j=1}^{r_j} \mu_{jn_j}^{m+2-2s_{jn_j}} C_{jn_j}. \tag{24.14}$$

In the case of "nonresonance" we can obtain the evaluation

$$\| x - x^{(m)} \| \leqslant \sum_{j=1}^{p} \sum_{n_j=1}^{r_j} \mu_{jn_j}^{m+1-s_{jn_j}} C_{jn_j}. \qquad (24.15)$$

The proof of inequalities (24.14) and (24.15) is carried out by the same method which was used to prove inequality (23.15).

Section 25. The Construction of the Asymptotic Solution for Other Sufficient Conditions

In constructing the asymptotic solution of system (17.1') when multiple elementary dividers are present, we have set a requirement that, for example, in the case when the root $\lambda_i(\tau)$ is associated with one elementary divider of multiplicity k_j from the coefficients of the system, condition (21.2) be satisifed in addition to the condition that they be infinitely differentiable with respect to τ.

In this section we specify an algorithm for constructing the asymptotic solutions of system (17.1') in the case when condition (21.2) is not satisfied on the segment $[0, L]$, i.e.,

$$\{c(\tau)\}_{r_j r_{j-1}+1} \equiv 0 \qquad (25.1)$$

for all $j=1, 2, \ldots, p(r_j=k_1+k_2+\ldots+k_j)$.

However, as shown in references 113 and 114, when condition (25.1) is satisfied, we must require that for the matrix $C(\tau)=T^{-1}(\tau)[A_1(\tau)T(\tau)-T'(\tau)]$ /154

$$\alpha_j(\tau) = \{c(\tau)\}_{r_j r_{j-1}+2} + \{c(\tau)\}_{r_j-1,\ r_{j-1}+1} \neq 0 \qquad (25.2)$$

for any $\tau \in [0, L]$ and for all $j=1, 2, \ldots, p$.

In addition to this, we shall assume that all the numbers k_j ($j=1, 2, \ldots, p$) which designate the multiplicity of the respective roots for the characteristic equation satisfy the condition

$$k_j > 2 \ (j = 1, 2, \ldots, p). \qquad (25.3)$$

If among the numbers k_j ($j=1, 2, \ldots, p$), numbers appear which are equal to 2, the algorithm presented below for the construction of an asymptotic solution to system (17.1') cannot be effective because it leads us to the necessity of

finding the solution of the Riccati equation. This case, which we have called "singular" is considered at the end of Section 25 (see the note on page 174).

Thus, if we assume that condition (25.3) is satisfied, we can prove the following theorem.

Theorem 4.7. If $A(\tau,\varepsilon), b(\tau,\varepsilon), \nu(\tau)$ on the segment $[0,L]$ have derivatives with respect to τ of all orders and the corresponding matrix elements $C(\tau)=T^{-1}(\tau)[A_1(\tau)T(\tau)-T'(\tau)]$ satisfy conditions (25.1) and (25.2), then the formal particular solution of system (17.1') in the case of "resonance" may be represented in the form

$$x = [U_1(\tau,\ \mu_1)h_1 + P(\tau,\ \varepsilon)]\,e^{i\theta(t,\varepsilon)} + \sum_{k=2}^{p} U_k(\tau,\ \mu_k)h_k, \tag{25.4}$$

$$\frac{dh_1}{dt} = [\lambda_1(\tau,\ \mu_1) - i\nu(\tau)]\,h_1 + z(\tau,\ \varepsilon), \tag{25.5}$$

$$\frac{dh_k}{dt} = \lambda_k(\tau,\ \mu_k)h_k,\ k = 2,\ 3,\ldots,\ p, \tag{25.6}$$

where $U_j(\tau.\mu_j)(j=1,\ 2\ \ldots,\ p), P(\tau,\varepsilon)$ are n-dimensional vectors and $\lambda_j(\tau.\mu_j), z(\tau,\varepsilon)$ are scalar functions which admit formal expansion, unlike (21.7), in powers of the parameter

$$\mu_j = \sqrt[k_j-1]{\varepsilon}\ (j = 1,\ 2,\ldots,\ p), \tag{25.7}$$

i.e.,

$$\begin{gathered} U_j(\tau,\ \mu_j) = \sum_{s=0}^{\infty} \mu_j^s U_{js}(\tau),\ \ \lambda_j(\tau,\ \mu_j) = \sum_{s=0}^{\infty} \mu_j^s \lambda_{js}(\tau), \\ P(\tau,\ \varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s P_s(\tau),\ \ z(\tau,\ \varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s z_s(\tau). \end{gathered} \tag{25.8}$$

Proof. If the vector x, determined by relationships (25.4)-(25.6), is the formal solution of the system, it must satisfy it. Therefore, if we substitute it into system (17.1') we obtain an identity. Let us require that in the identity obtained in this manner the coefficients in front of h_j and the free terms be equal. As a result of this we shall have /155

$$[A(\tau,\ \varepsilon)-\lambda_j(\tau,\ \mu_j)E]U_j(\tau,\ \mu_j)=\varepsilon U_j'(\tau,\ \mu_j), \tag{25.9}$$

$$[A(\tau,\ \varepsilon)-iv(\tau)E]P(\tau,\ \varepsilon)=U_1(\tau,\ \mu_1)z(\tau,\ \varepsilon)+\varepsilon(P'(\tau,\ \varepsilon)-b(\tau,\ \varepsilon)). \tag{25.10}$$

(1) To determine the coefficients of the vector $U_j(\tau,\mu_j)$ and the functions $\lambda_j(\tau,\mu_j)$ we shall make use of relationship (25.9). By comparing the coefficients in front of the same powers of parameter ε in this relationship, we obtain

$$[A_0(\tau)-\lambda_{j_0}(\tau)E]U_{j_0}(\tau)=0, \tag{25.11}$$

$$[A_0(\tau)-\lambda_{j_0}(\tau)E]U_{js}(\tau)=\sum_{i_0=1}^{s-1}U_{ji_0}(\tau)\lambda_{js-i_0}(\tau)+H_{js}(\tau),$$

$$(s=1,\ 2,\ldots;\qquad j=1,\ 2,\ldots,\ p), \tag{25.12}$$

where

$$H_{js}(\tau)=U_{js+1-k_j}'(\tau)-\sum_{k=1}^{\left[\frac{s}{k_j-1}\right]}A_k(\tau)U_{js-k(k_j-1)}\ (\tau) \tag{25.13}$$

$$(j=1,\ 2,\ldots,\ p).$$

In the future we shall require the vector

$$Q_{js}(\tau)=T^{-1}(\tau)U_{js}(\tau)\ (s=0,\ 1,\ldots;\qquad j=1,\ 2,\ldots,\ p) \tag{25.14}$$

and the vector of dimensionality k_i (i=1, ..., p)

$$Q_{jis}(\tau)=\begin{bmatrix}\{q_{js}\}_{r_{i-1}+1}\\ \{q_{js}\}_{r_{i-1}+2}\\ \cdot\\ \cdot\\ \cdot\\ \{q_{js}\}_{r_i}\end{bmatrix}(r_i=k_1+k_2+\ldots+k_i;\ i=1,\ 2,\ldots,\ p), \tag{25.15}$$

formed from the respective components of vector $Q_{js}(\tau)$.

Then, on the basis of (20.1), equation (25.11) may be written in the form /156

$$[W_i(\tau)-\lambda_{j_0}(\tau)E]Q_{ji_0}(\tau)=0\ (i,j=1,\ 2,\ldots,\ p). \tag{25.16}$$

Since /156

$$\det [W_i(\tau) - \lambda_{j_0}(\tau) E] \neq 0, \; i \neq j, \tag{25.17}$$

for all $\tau \in [0, L]$, it follows from equation (25.16) that

$$Q_{ji_0}(\tau) = 0 \; (i \neq j; \quad i, \; j = 1, \; 2, \ldots, \; p). \tag{25.18}$$

When i=j, equation (25.16) assumes the form

$$I_j Q_{jj_0}(\tau) = 0 \; (j = 1, \; 2, \ldots, \; p), \tag{25.19}$$

where

$$I_j = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & 0 & \ldots & 0 \end{bmatrix}. \tag{25.20}$$

From this we find

$$\{q_{j_0}(\tau)\}_{r_{j-1}+n_j} = 0 \; (n_j = 2, \; 3, \ldots, \; k_j; \qquad j = 1, \ldots, \; p). \tag{25.21}$$

Since the component $\{q_{j_0}(\tau)\}_{r_{j-1}+1}$ is arbitrary, we set it equal to

$$\{q_{j_0}(\tau)\}_{r_{j-1}+1} = 1 \; (j = 1, \; 2, \ldots, \; p). \tag{25.22}$$

Thus the vector $Q_{j_0}(\tau)$ is determined. If we know this vector, we can use equation (25.14) to find the vector $U_{j_0}(\tau)$ (j=1, 2, ..., p).

To determine the vectors $U_{js}(\tau)$ and the function $\lambda_{js}(\tau)$, we shall use equation (25.12), which may be written in the following manner

$$[W_i(\tau) - \lambda_{j_0}(\tau) E] Q_{jis}(\tau) = \sum_{k=0}^{s-1} Q_{jik}(\tau) \lambda_{js-k}(\tau) + \bar{H}_{jis}(\tau) \tag{25.23}$$

$$(j, \; i = 1, \; 2, \ldots, \; p; \qquad s = 1, \; 2, \ldots),$$

where

$$\bar{H}_{js}(\tau) = T^{-1}(\tau) H_{js}(\tau). \tag{25.24}$$

Then by taking into account relationship (25.17), we have /157

$$Q_{jis}(\tau) = [W_i(\tau) - \lambda_{j0}(\tau) E]^{-1} \left[\sum_{k=0}^{s-1} Q_{jis}(\tau) \lambda_{js-k}(\tau) + \widetilde{H}_{jis}(\tau) \right] \tag{25.25}$$

$$(i \neq j; \quad i, j = 1, 2, \ldots, p; \quad s = 1, 2, \ldots).$$

We note that because

$$\bar{H}_{ji}(\tau) \equiv 0 \quad \text{when} \quad 1 \leqslant s \leqslant k_j$$

we have

$$Q_{jis}(\tau) \equiv 0, \; 1 \leqslant s \leqslant k_j - 2; \quad i \neq j. \tag{25.26}$$

We must now investigate equation (25.23) for j=i. In this case it assumes the form

$$I_j Q_{jjs}(\tau) = \sum_{k=0}^{s-1} Q_{jik}(\tau) \lambda_{js-k}(\tau) + \bar{H}_{jjs}(\tau) \tag{25.27}$$

$$(s = 1, 2, \ldots; \quad j = 1, 2, \ldots, p).$$

The first components of vectors $Q_{jjs}(\tau)$ in equation (25.27) remain arbitrary. For simplicity we shall let

$$\{q_{js}(\tau)\}_{r_{j-1}+1} = 0 \quad (r_j = k_1 + k_2 + \ldots + k_j, \; j = 1, 2, \ldots, p, \tag{25.28}$$

$$s = 1, 2, \ldots).$$

The remaining components of vectors $Q_{jjs}(\tau)$ contain the unknown functions $\lambda_{js}(\tau)$, which can be determined from the last equation of system (25.27):

$$\sum_{k=0}^{s-1} \{q_{jk}(\tau)\}_{r_j} \lambda_{js-k}(\tau) + \{\widetilde{h}_{js}(\tau)\}_{r_j} = 0 \; (s = 1, 2, \ldots; \; j = 1, 2, \ldots, p). \tag{25.29}$$

While investigating equation (25.29), we shall proceed in exactly the same manner as in the investigation of equation (21.32), i.e., in place of equation (25.29) we shall examine the equivalent first scalar equation in the new system obtained from system (25.27) when the left side of the latter is multiplied k_j-1 times by the matrix I_j. As a result of this we shall have

$$\sum_{i_{k_j-1}=k_j-1}^{s-1} \sum_{i_{k_j-2}=k_j-2}^{i_{k_j-1}-1} \cdots \sum_{i_0=0}^{i_1-1} Q_{jji_0}(\tau) \lambda_{ji_1-i_0}(\tau) \lambda_{ji_2-i_1}(\tau) \ldots \lambda_{js-i_{k_j-1}} + \tag{25.30}$$

$$+ F_{jjs}(\tau) = 0 \; (s = 1, 2, \ldots),$$

where

$$F_{jjs}(\tau)=\sum_{i_{k_j-1}=2k_j-2}^{s-1}\sum_{i_{k_j-2}=2k_j-3}^{i_{k_j-1}-1}\cdots\sum_{i_1=k_j-1}^{i_2-1}\bar{H}_{jji_1}(\tau)\lambda_{ji_2-i_1}(\tau)\ldots\lambda_{js-ik_j-1}(\tau)+$$
$$+\sum_{i_{k_j-1}=2k_j-3}^{s-1}\sum_{i_{k_j-2}=2k_j-4}^{i_{k_j-1}-1}\cdots\sum_{i_2=k_j-1}^{i_3-1}I_j\bar{H}_{jji_2}(\tau)\lambda_{ji_3-i_2}(\tau)\ldots\lambda_{js-i_{k_j-1}}(\tau)+\ldots+ \qquad (25.31)$$
$$+\sum_{i_{k_j-1}=k_j-1}^{s-1}I_j^{k_j-2}\bar{H}_{jji_{k_j-1}}(\tau)\lambda_{js-i_{k_j-1}}(\tau)+I_j^{k_j-1}\bar{H}_{jjs}(\tau)$$
$$(s=1,\ 2,\ldots,\ j=1,\ 2,\ldots,\ p).$$

If we let $s=k_j-1$ in equation (25.30) (for $1\leq s\leq k_1-2$, it transforms into an identity according to (25.13)), we have

$$F_{jjk_j-1}(\tau)=0 \qquad (25.32)$$

or, by taking (25.31), into account we have

$$I_j^{k_j-1}H_{jjk_j-1}(\tau)=0,\ j=1,\ 2,\ldots,\ p, \qquad (25.33)$$

where

$$\bar{H}_{jjk_j-1}(\tau)=C_j(\tau)Q_{j0}(\tau), \qquad (25.34)$$

The quantity $C_j(\tau)$ is a rectangular matrix of dimensions $(k_j \times n)$, which is formed from the corresponding elements of the matrix $C(\tau)$.

On the basis of (25.1), equation (25.33) becomes an identity.

Now let $s=k_j$. Then equation (25.30) is transformed into the equation

$$Q_{jj_0}(\tau)[\lambda_{j_1}(\tau)]^{k_j}+F_{jjk_j}(\tau)=0, \qquad (25.35)$$

in which

$$F_{jjk_j}(\tau)=I_j^{k_j-2}\bar{H}_{jjk_j-1}(\tau)\lambda_{j_1}(\tau)+I_j^{k_j-1}(Q'_{jj_1}(\tau)+C_j(\tau)Q_{j_1}(\tau)). \qquad (25.36)$$

As follows from equations (25.26) and (25.27), the vector $Q_{j1}(\tau)$ has components which are equal to zero, except for the component

$$\{q_{j_1}(\tau)\}_{r_{j-1}+2}=\lambda_{j_1}(\tau)\ (j=1,\ 2,\ldots,\ p). \qquad (25.37)$$

Then, by taking into account condition (25.3) we obtain the following from equation (25.35):

$$[\lambda_{j_1}(\tau)]^{k_j} + \alpha_j(\tau)\lambda_{j_1}(\tau) = 0 \ (j = 1, 2, \ldots, p) \tag{25.38}$$

(when $k_j=2$, the first equation in system (23.35) transforms into the Riccati equation). /159

From this we have

$$\lambda_{j_1}^{(1)}(\tau) = 0, \ \lambda_{j_1}^{(2)}(\tau) = \sqrt[k_j-1]{-\alpha_j(\tau)}, \ j = 1, 2, \ldots, p. \tag{25.39}$$

When $s=k_j+1$, we have the following from equation (25.30):

$$k_j[\lambda_{j_1}(\tau)]^{k_j-1}\lambda_{j_2}(\tau)Q_{jj_0}(\tau) + [\lambda_{j_1}(\tau)]^{k_j}Q_{jj_1}(\tau) + F_{jjk_j+1}(\tau) = 0$$
$$(j = 1, 2, \ldots, p), \tag{25.40}$$

where

$$F_{jjk_j+1}(\tau) = I_j^{k_j-1}(Q'_{jj_2}(\tau) + C_j(\tau)Q_{j_2}(\tau)) + I_j^{k_j-2}\overline{H}_{jjk_j-1}(\tau)\lambda_{j_2}(\tau) + \tag{25.41}$$
$$+ I_j^{k_j-3}\overline{H}_{jjk_j}(\tau)\lambda_{j_1}(\tau) (j = 1, 2, \ldots, p).$$

Taking the first equation in system (25.40) and noting that

$$\{q_{j2}(\tau)\}_{r_{j-1}+2} = \lambda_{j2}(\tau), \tag{25.42}$$

we shall have

$$\lambda_{j2}(\tau)[k_j(\lambda_{j1}(\tau))^{k_j-1} + \alpha_j(\tau)] + \beta_{j2}(\tau) = 0, \tag{25.43}$$

where

$$\beta_{j2}(\tau) = \sum_{i=1}^{n}{}^{*}\{C(\tau)\}_{r_j,i}\{q_{ji}(\tau)\}_2 + \{c(\tau)\}_{r_j-3,\ r_{j-1}-2}(\lambda_{j_1}(\tau))^2 \tag{25.44}$$

(the term corresponding to $i=r_{j-1}+2$ is dropped in the sum Σ^*; the components of the vector $Q_{j2}(\tau)$ contained in the sum Σ^* are expressed in terms of the function $\lambda_{j1}(\tau)$, which is already known).

Then, from equation (23.43) we find

$$\lambda_{j2}(\tau) = -\frac{\beta_{j2}(\tau)}{k_j\lambda_{j1}(\tau) + \alpha_j(\tau)} \quad (j=1, 2,\ldots, p). \tag{25.45}$$

The same method can be used to determine all the subsequent functions $\lambda_{js}(\tau)$(s=3, 4, ...). Thus, for example, by letting $s=k_j+m-1$ in (23.30) (m is any natural number) we have

$$\lambda_{jm}(\tau) = -\frac{\beta_{jm}(\tau)}{k_j(\lambda_{j1})^{k_j-1} + \alpha_j} \quad (j=1,\ 2,\ldots, p), \tag{25.46}$$

where /160

$$\beta_{jm}(\tau) = \sum_{i=1}^{n}{}^{*}\{c(\tau)\}_{r_j,i}\{q_{jm}(\tau)\}_i + \{c(\tau)\}_{r_j,r_{j-1}+2}\{\bar{h}_{jm}(\tau)\}_{r_{j-1}+1} + \\ + \gamma_{jm}(\tau),\ j=1,\ 2,\ldots, p, \tag{25.47}$$

$\gamma_{jm}(\tau)$ is the first component of the vector $F_{jjm}(\tau)$:

$$F_{jjm}(\tau) = I_j^{k_j-1}\Big[Q'_{jjm}(\tau) - \sum_{k=2}^{\left(\frac{k_j+m-1}{k_j-1}\right)} c_{jk}(\tau)\,Q_{jk_j+m-1-k(k_j-1)}(\tau) + \\ + \sum_{i_{k_j-1}=k_j}^{k_j+m-2} I_j^{k_j-2}\bar{H}_{jji_{k_j-1}}(\tau)\,\lambda_{jm+k_j-1-ik_j-1}(\tau) + \ldots + \\ + \sum_{i_{k_j-1}=2k_j-2}^{k_j+m-2}\ \sum_{i_{k_j-2}=2k_j-3}^{i_{k_j-1}-1}\ldots\sum_{i_1=k_j-1}^{i_2-1}\bar{H}_{jji_1}(\tau)\,\lambda_{ji_2-i_1}(\tau)\ldots\lambda_{jk_j+m-1-i_{k_j-1}}(\tau)\Big], \tag{25.48}$$
$$(j=1,\ 2,\ldots, p),$$

where $C_{jk}(\tau)$ is a rectangular matrix of the dimensions $(k_j \times n)$ formed from the elements of the matrix $T^{-1}(\tau)\,A_k(\tau)\,T(\tau)$ $\left(k=2,\ 3,\ldots,\left\lceil\frac{k_j+m-1}{k_j-1}\right\rceil\right)$.

It should be noted that the denominator in equations (25.45) and (25.46), according to (25.2) and (25.39), does not become equal to zero for all $\tau\in[0,L]$. Thus, the method we specified makes it possible to determine the functions

$\lambda_{j1}(\tau)$, ..., $\lambda_{jm}(\tau)$ (j=1, 2, ..., p) for any natural m. Knowing these functions we can use (25.27) to determine the vectors $Q_{jj1}(\tau), Q_{jj2}(\tau)$, ..., $Q_{jjm}(\tau)$ quite easily.

(2) Now let us find the coefficients of vector $P(\tau, \varepsilon)$ and of the function $z(\tau, \varepsilon)$.

For this purpose, by separating the coefficients of order ε^s in relationship (25.10) we have

$$[A_0(\tau) - i\nu(\tau) E] P_s(\tau) = \sum_{k=0}^{s} U_{1k(k_1-1)}(\tau) z_{s-k}(\tau) + \\ + P'_{s-1}(\tau) - b_{s-1}(\tau) - \sum_{k=1}^{s} A_k(\tau) P_{s-k}(\tau) (s = 0, 1, \ldots). \tag{25.49}$$

We consider the vector /161

$$R_s(\tau) = T^{-1}(\tau) P_s(\tau) \quad (s = 0, 1, 2, \ldots). \tag{25.50}$$

Then, when s=0 equation (25.49) may be written in the form

$$[W_0(\tau) - i\nu(\tau) E] R_0(\tau) = Q_{10}(\tau) z_0(\tau),$$

or

$$[W_{j0}(\tau) - i\nu(\tau) E] R_{j0}(\tau) = Q_{1j0}(\tau) z_0(\tau) \quad (j = 1, 2, \ldots, p). \tag{25.51}$$

From this, according to (19.4), we have

$$R_{j0}(\tau) = 0 \quad (j = 2, 3, \ldots, p). \tag{25.52}$$

When j=1, the function $i\nu(\tau)$ becomes equal to $\lambda_{10}(\tau)$. Therefore, it follows from equation (25.51) that

$$\{r_0(\tau)\}_2 = z_0(\tau), \quad \{r_0(\tau)\}_{r_1} = 0, \quad 3 \leqslant r_1 \leqslant k_1. \tag{25.53}$$

The first components of vectors $R_s(\tau)$ (s=1, 2, ...) are set equal to zero because they are arbitrary.

We let s=1, which gives us

$$[W_0(\tau) - i\nu(\tau) E] R_1(\tau) = Q_{10}(\tau) z_1(\tau) + Q_{1k_1-1}(\tau) z_0(\tau) + \\ + c(\tau) R_0(\tau) + R'_0(\tau) + G_0(\tau), \tag{25.54}$$

where

$$G_0(\tau) = -T^{-1}(\tau) b_0(\tau). \quad (25.55)$$

According to (18.3), equation (25.54) may be written in the form

$$[W_{j0}(\tau) - i\nu(\tau) E] R_{j1}(\tau) = Q_{1j0}(\tau) z_1(\tau) + Q_{1/k_1-1}(\tau) z_0(\tau) + \\ + c_j(\tau) R_0(\tau) + R'_{j0}(\tau) + G_{j0}(\tau) \quad (j = 1, 2, \ldots, p). \quad (25.56)$$

From this we have

$$R_{j1}(\tau) = [W_{j0}(\tau) - i\nu(\tau) E]^{-1} [Q_{1/k_1-1}(\tau) z_0(\tau) + \\ + c_j(\tau) R_0(\tau) + G_{j0}(\tau)] \quad (j = 2, 3, \ldots, p). \quad (25.57)$$

When j=1, equation (25.56) may be written in the following manner:

$$\{r_1(\tau)\}_2 = z_1(\tau) + \{c(\tau)\}_{12} \{r_0(\tau)\}_2 + \{g_0(\tau)\}_1, \\ \{r_1(\tau)\}_l = \{q_{1k_1-1}(\tau)\}_{l-1} z_0(\tau) + \{c(\tau)\}_{l-1,2} \{r_0(\tau)\}_2 + \{g_0(\tau)\}_l \\ (3 \leqslant l \leqslant k_1), \quad (25.58)$$

$$\{q_{1k_1-1}(\tau)\}_k z_0(\tau) + \{c(\tau)\}_{k_1 2} \{r_0(\tau)\}_2 + \{g_0(\tau)\}_{k_1} = 0. \quad (25.59)$$

Then, from equation (25.59), we find /162

$$z_0(\tau) = -\frac{\{g_0(\tau)\}_{k_1}}{\alpha_1(\tau)}. \quad (25.60)$$

The same method may be used to determine all the subsequent coefficients $R_s(\tau), Z_s(\tau)$ (s=1, 2, ...). For example, let us find $R_m(\tau), Z_m(\tau)$, assuming that $R_0(\tau), R_1(\tau), \ldots, R_{m-1}(\tau), Z_0(\tau), Z_1(\tau), \ldots, Z_{m-1}(\tau)$ are already known. To do so we let s=m in equation (25.49). We shall have

$$[W(\tau) - i\nu(\tau) E] R_m(\tau) = Q_{10}(\tau) Z_m(\tau) + G_{m-1}(\tau), \quad (25.61)$$

where

$$G_{m-1}(\tau) = T^{-1}(\tau) \left[\sum_{i=1}^{m} (U_{1i(k_1-1)}(\tau) Z_{m-i}(\tau) - A_i(\tau) P_{m-i}(\tau)) + \\ + P'_{m-1}(\tau) - b_{m-1}(\tau) \right]. \quad (25.62)$$

From equation (25.61) we find

$$R_{jm}(\tau) = [W_{j0}(\tau) - i\nu(\tau)E]^{-1} G_{jm-1}(\tau) \quad (j = 2, 3, \ldots, p), \tag{25.63}$$

$$\{r_m(\tau)\}_2 = Z_m(\tau) + \{g_{m-1}(\tau)\}_1,$$
$$\{r_m\}_l = \{g_{m-1}(\tau)\}_{l-1} \quad (3 \leqslant l \leqslant k_1),$$
$$\{g_{m-1}(\tau)\}_{k_1} = 0. \tag{25.64}$$

The last equation determines the function $z_{m-1}(\tau)$.

Now by letting s=m+1 in relationship (25.49) and repeating the arguments used in determining the function $z_0(\tau)$, we obtain

$$z_m(\tau) = \frac{\{g_m(\tau)\}_{k_1} + \sum_{i=3}^{n} \{c(\tau)\}_{k_1 i} \{r_m(\tau)\}_i + \{c(\tau)\}_{k_1 2} \{g_{m-1}(\tau)\}_2}{\alpha_1(\tau)}, \tag{25.65}$$

where $\{g_m(\tau)\}_{k_1}$ is the k_1-th component of the vector

$$G_m(\tau) = \sum_{i=2}^{m+1} (Q_{1i(k_1-1)}(\tau) z_{m+1-i}(\tau) - T^{-1}(\tau) A_i(\tau) P_{m+1-i}(\tau)) - \tag{25.66}$$
$$- T^{-1}(\tau)(b_m(\tau) - P'_m(\tau)).$$

Thus, by specifying a method for obtaining the coefficients for formal series (25.8) we have proven the given theorem.

1. We note that the theorem which we have proved makes it possible to obtain not only the particular solution of system (17.1') but also its general solution. Indeed, according to (25.39), the functions $\lambda_{j1}(\tau)$ (it can be shown that this also includes all the subsequent functions $\lambda_{js}(\tau)$, s=2, 3, ...) have k_j different values. Consequently, for differential equations (25.5) and (25.6), we can construct k_j $\left(\sum_{j=1}^{p} k_j = n;\ j = 1, 2, \ldots, p\right)$ particular linearly independent solutions, which shows that a general solution can also be constructed. If we substitute it in place of h_j (j=1, 2, ..., p) into the particular solution (25.4) we shall obtain the formal general solution of system (17.1'). /163

For this purpose we apply operator $T(t-s)$ to both sides of equality (29.15) and integrate the result from t_0 to t:

$$\int_{t_0}^{t} T(t-s)\,y'(s)\,ds = \int_{t_0}^{t} T(t-s)\,Ay(s)\,ds + \int_{t_0}^{t} T(t-s)\,B(s)y(s)\,ds. \tag{29.16}$$

Since the following equality is valid for $s\in[t_0, t]$:

$$\frac{d}{ds}\,[T(t-s)\,y(s)] = -T(t-s)\,Ay(s) + T(t-s)\,y'(s),$$

we find from relationship (29.16) that

$$y(t) = \int_{t_0}^{t} T(t-s)\,B(s)\,y(s)\,ds. \tag{29.17}$$

In view of (29.14), $T(t-t_0)y(t_0)=0$, $0\le t_0\le t<\infty$.)

Let us assume that $\max_{\|}y(s)_{\|}=m_t$. Then according to (29.17) there follows the inequality:

$$m_t < Mm_t(t-t_0) \tag{29.18}$$

for any $t\ge t_0$.

However, it is clear that for t, which is sufficiently close to t_0, $t_0\le t\le t_1$, where $M(t_1-t_0)<1$, inequality (29.18) is possible only when $m_t=0$. Thus we reach the conclusion that the solution $y(t)$ which we have constructed must be identically equal to zero over the segment $[t_0, t_1]$.

Now, if instead of expression (29.17) we investigate the equivalent relationship

$$y(t) = \int_{t_1}^{t} T(t-s)\,B(s)\,y(s)\,ds; \quad y(t)\equiv 0 \quad \text{for} \quad t\in[t_0, t_1],$$

then by using the approach described above we can see that $y(t)\equiv 0$ for any $t\ge t_0$, i.e., the operators $V(t, t_0)$ and $V_1(t, t_0)$ are identical.

Thus we have shown that equation (29.1) with the given initial condition /190

$$x(t_0) = x_0 \in D(A) \quad (t_0 \geqslant 0) \tag{29.19}$$

is differentiable in respect to t. Moreover, in view of the properties of expression $\frac{d}{dt}W_n(t,t_0)x_0$, the function $V(t,t_0)x_0$ is continuously differentiable with respect to t.

Since in this case it is obvious that expression $B(t)V(t,t_0)x_0(x_0\in D(A))$ is also continuously differentiable, we can use Lemma 5.1 and differentiate both sides of equality (29.10) with respect to t. As a result we obtain

$$\begin{aligned}\frac{d}{dt}[V(t,t_0)x_0] &= AT(t-t_0)x_0 + B(t)V(t,t_0)x_0 + \\ &+ A\int_{t_0}^{t} T(t-s)B(s)V(s,t_0)x_0 ds = \\ &= [A+B(t)]V(t,t_0)x_0 \quad (x_0\in D(A)).\end{aligned} \tag{29.13}$$

Thus we become convinced that the function

$$x(t) = V(t,t_0)x_0$$

satisfies equation (29.1) and condition $x(t_0)=x_0$. We also note that the property of operator $V(t,t_0)$ to keep the domain D invariant follows from equality (29.13):

$$V(t,t_0)D\subseteq D.$$

Let us now show that the operator $V(t,t_0)$ which satisfies the condition $V(t_0,t_0)=I$ is unique.

Let us assume that yet another operator $V_1(t,t_0)$ exists such that $V_1(t_0,t_0)=I$ and the function $x_1(t)=V_1(t,t_0)x_0$ is also a solution of equation (29.1). Then it is obvious that equation (29.1) also has a solution of the form

$$y(t) = [V(t,t_0) - V_1(t,t_0)]x_0,$$

and that y(t) has all the properties described above, as well as: /189

$$y(t_0) = [V(t_0,t_0) - V_1(t_0,t_0)]x_0 = 0. \tag{29.14}$$

Let us examine the solution y(t) which has been constructed for the equation

$$\frac{dy}{dt} = [A+B(t)]y. \tag{29.15}$$

Let us first establish that expressions (29.11) are differentiable with respect to t.

It is obvious that the function $W_0(t,t_0)x=V_0(t,t_0)x_0$ for $x_0\in D(A)$ is continuously differentiable with respect to t. Let us assume that this is also true for $W_n(t,t_0)x_0$. Then it is easy to show that the element $B(t)W_n(t,t_0)x_0$ ($n=0, 1, 2, \ldots$) will also be continuously differentiable with respect to t and consequently, according to Lemma 5.1, the function $W_{n+1}(t,t_0)x_0$ will have the same properties (see eq. (29.7)).

$$\frac{d}{dt}W_0(t,t_0)x_0 = T(t-t_0)Ax_0,$$

$$\frac{d}{dt}W_n(t,t_0)x_0 = T(t-t_0)B(t_0)W_{n-1}(t_0,t_0)x_0 +$$

$$+\int_{t_0}^{t} T(t-s)B'(s)W_{n-1}(s,t_0)x_0 ds +$$

$$+\int_{t_0}^{t} T(t-s)B(s)\frac{dW(s,t_0)}{dt}x_0 ds \qquad (n=1,2,\ldots).$$

From this the following estimate follows:

$$\left\|\frac{d}{dt}W_0(t,t_0)x_0\right\| \leqslant \|Ax_0\|,$$

$$\left\|\frac{d}{dt}W_n(t,t_0)x_0\right\| \leqslant M^n\frac{(t-t_0)^{n-1}+(t-t_0)^n}{(n-1)!}\alpha, \qquad (29.12)$$

where

$$\alpha = \|Ax_0\| + \|x_0\|.$$

Thus, in view of inequality (29.12) we can maintain that the series

$$\sum_{n=0}^{\infty}\frac{d}{dt}W_n(t,t_0)x_0$$

converges uniformly with respect to t in each finite interval. Consequently the function

$$x(t) = V(t,t_0)x_0 = \sum_{n=0}^{\infty}W_n(t,t_0)x_0 \quad (x_0\in D(A))$$

$$V(t, t_0) = \sum_{n=0}^{\infty} W_n(t, t_0). \tag{29.8}$$

Let us show that our preceding constructions are meaningful and that operator $V(t, t_0)$ indeed exists.

First of all it is clear that for any $x \in \mathfrak{X}$ the vector function $B(t)V_0(t, t_0)x \equiv B(t)T(t-t_0)x$ is continuous with respect to t, and consequently, according to Lemma 5.1, the function $W_1(t, t_0)x$ exists and is also continuous with respect to t (obviously this is also true for the function $V_1(t, t_0)x$). By the method of induction we obtain analogous statements for any vector $W_n(t, t_0)x$, i.e., for any approximation $V_n(t, t_0)x$.

Since $\|W_0(t, t_0)\| \equiv \|T(t-t_0)\| \leqslant 1$ $(0 \leqslant t_0 \leqslant t)$, , it is obvious that

$$\|W_n(t, t_0)\| \leqslant M^n \frac{(t-t_0)^n}{n!}, \tag{29.9}$$

where

$$M = \max_{0 \leqslant t \leqslant \frac{L}{\varepsilon}} [\|B(t)\|, \|B'(t)\|].$$

Consequently the series (29.8) (or the sequence (29.6)) converges (in the sense of operators' norm) uniformly with respect to t (in each finite interval) to the bounded operator $V(t, t_0)$, which is strongly continuous with respect to t in the interval (t_0, ∞). /187

According to (29.8) and (29.9), the following estimate exists:

$$\|V(t, t_0)\| \leqslant e^{M(t-t_0)}.$$

Since $V_n(t_0, t_0) = I$ for any n, we also have $V(t_0, t_0) = I$. Finally by approaching the limit in equations (29.6), we obtain

$$V(t, t_0)x = T(t-t_0)x + \int_{t_0}^{t} T(t-s)B(s)V(s, t_0)x\,ds. \tag{29.10}$$

We shall now show that the function

$$x(t) = V(t, t_0)x_0 \quad (x_0 \in D(A)) \tag{29.11}$$

is a solution of equations (29.1).

Let us now investigate the solvability of equation (29.1). The existence and uniqueness of the solution to the equation (29.1) are established by the following theorem.

Theorem 5.1. Let the operator A satisfy conditions (S_1) and the operator B(t) satisfy conditions (S_2)*.

Then there exists a unique bounded operator $V(t,t_0)\left(0\leq t_0\leq\tau\leq\frac{L}{\varepsilon}\right)$ such that $V(t_0,t_0)=I$ and the function $x(t)=V(t,t_0)x_0\,(x_0\in D(A))$ is the solution of equation (29.1) satifying the condition

$$x(t_0)=x_0.$$

In addition, the operator function $V(t,s)f$ for $f\in D(A)$, when the parameter s varies in the limits $0\leq s\leq t\leq\frac{L}{\varepsilon}$, is continuously differentiable with respect to s /186 and satisfies the equation

$$\frac{dV(t,s)}{ds}f=-V(t,s)[A+B(s)]f,\quad f\in D(A).$$

Proof. To construct the operator $V(t,t_0)$ we use the method of successive approximations and introduce the following designations:

$$\begin{gathered}V(t,t_0)=T(t-t_0),\\ V_n(t,t_0)x=T(t-t_0)x+\int_{t_0}^{t}T(t-s)B(s)V_{n-1}(s,t_0)x\,ds\\ (x\in\mathfrak{X},\quad n=1,2,\ldots).\end{gathered}\tag{29.6}$$

Also, if we let

$$\begin{gathered}W_0(t,t_0)=V_0(t,t_0),\\ W_n(t,t_0)x=[V_n(t,t_0)-V_{n-1}(t,t_0)]x=\\ =\int_{t_0}^{t}T(t-s)B(s)W_{n-1}(s,t_0)x\,ds,\end{gathered}\tag{29.7}$$

the unknown operator $V(t,t_0)$ may be represented in the form

*
See footnote on page 192.

Since $T(t+\Delta-\sigma)\varphi(\sigma)$ is a continuous function of its variables and $T(0)=I$, we have

$$\frac{1}{\Delta}\int_{t}^{t+\Delta} T(t+\Delta-\sigma)\varphi(\sigma)\,d\sigma \to \varphi(t), \qquad \text{if} \quad \Delta \to 0.$$

Furthermore, as shown above, the function $g(t,s)$ is differentiable, i.e., the limit of relationship (29.5) exists when $\Delta\to 0$ and consequently $\int_{s}^{t} T(t-\sigma)\varphi(\sigma)\,d\sigma \in D(A)$ (here we make use of the well-known properties of operator $T(t)$(ref. 102)):

$$\lim_{\eta\to 0}\frac{T(t+\eta)-I}{\eta}x = Ax).$$

In this manner we also obtain the second equality (29.4'): /185

$$\frac{dg(t,s)}{dt} = \varphi(t) + A\int_{s}^{t} T(t-\sigma)\varphi(\sigma)\,d\sigma, \quad 0 \leqslant s < t < \infty.$$

In conclusion we note that in view of (29.3) and in view of the properties of function $g(t,s)$ established above, it is easy to see that the function $g(t,s)$ is also continuously differentiable with respect to the variable s, $0\leq s<t<\infty$, and that the derivative $\frac{dg(t,s)}{ds}$ will be continuous in t.

Supplement to Lemma 5.1. If the function $\varphi(t)$ $(t\in[0,\infty])$ assigned in Lemma 5.1 also depends on the real parameter s $(0\leq s<t<\infty)$, i.e., $\varphi=\varphi(t,s)$ while the function $\varphi(t,s)$ is continuously differentiable with respect to this parameter s and its derivative $\frac{d\varphi(t,s)}{ds}$ is also continuous with respect to t, then the function

$$g(t,s) = \int_{s}^{t} T(t-\sigma)\varphi(\sigma,s)\,d\sigma$$

will also have the same properties. In this case

$$\frac{dg(t,s)}{ds} = -T(t-s)\varphi(s,s) + \int_{s}^{t} T(t-\sigma)\frac{d\varphi(\sigma,s)}{ds}\,d\sigma.$$

The validity of this proposition follows from the results of Lemma 5.1.

$$g(t,s)=\int_s^t T(t-\sigma)\varphi(\sigma)\,d\sigma \qquad (0\leqslant s<t<\infty)$$

exists and equality (29.3) is valid.

The function g(t,s) is continuous with respect to t, which is easy to see from the following inequalities:

$$\|g(t+\Delta,s)-g(t,s)\|\leqslant\int_s^t\|T(\sigma)[\varphi(t+\Delta-\sigma)-\varphi(t-\sigma)]\|\,d\sigma+$$
$$+\int_t^{t+\Delta}\|T(\sigma)\varphi(t+\Delta-\sigma)\|\,d\sigma\leqslant\int_s^t\|\varphi(t+\Delta-\sigma)-\varphi(t-\sigma)\|\,d\sigma+$$
$$+\int_t^{t+\Delta}\|\varphi(t+\Delta-\sigma)\|\,d\sigma.$$

Indeed, since $\varphi(t)$ is continuous with respect to $\tau\in[0,\infty]$, obviously $\|g(t+\Delta,s)-g(t,s)\|\to 0$, when $\Delta\to 0$, i.e., the function g(t,s) is continuous with respect to t. /184

Furthermore, in view of the propositions in the lemma, the vector $T(\sigma)\varphi'(t-\sigma)$ is continuous over σ when $\sigma\in[0,t]$; therefore, according to the second part of equality (29.3), we are permitted to write

$$\frac{dg(t,s)}{dt}=T(t-s)\varphi(s)+\int_0^{t-s}T(\sigma)\varphi'(t-\sigma)\,d\sigma=$$
$$=T(t-s)\varphi(s)+\int_s^t T(t-\sigma)\varphi'(\sigma)\,d\sigma,$$

from which it is clear that the derivative $\frac{dg(t,s)}{dt}$ is also continuous over t.

On the other hand, by using the representation

$$g(t,s)=\int_s^t T(t-\sigma)\varphi(\sigma)d\sigma,$$

we consider the following relationship

$$\frac{g(t+\Delta,s)-g(t,s)}{\Delta}=\frac{T(\Delta)-I}{\Delta}\int_s^t T(t-\sigma)\varphi(\sigma)\,d\sigma+$$
$$+\frac{1}{\Delta}\int_t^{t+\Delta}T(t+\Delta-\sigma)\varphi(\sigma)\,d\sigma \qquad (29.5)$$

when $\Delta\to 0$.

The existence and uniqueness of the solution to equation (29.1) with a given initial condition were investiaged by P. Phillips in reference 134. However, for the sake of making our presentation complete we shall reproduce these results here.

We note first that when conditions (S_1) are satisfied there is a bounded operator $T(t), \tau\in[0,\infty]$ which has the following properties:

$$\| T(t) \| \leqslant 1, \qquad T(t_1 + t_2) = T(t_1)\, T(t_2),$$

$T(t)$ is highly continuous with respect to t in $[0,\infty], T(0)=I$. In addition, for /183 any $x\in D$ the vector function $T(t)x$ is continuously differentiable and satisfies the equation

$$\frac{d[T(t)x]}{dt} = AT(t)x = T(t)Ax, \qquad x \in D. \tag{29.2}$$

In the future we shall make use of the following lemma.

Lemma 5.1. If the function $\varphi(t)$, which is determined in $[0,\infty]$ and which has values in the space $\mathfrak{X}$, is continuously differentiable, the function

$$g(t,s) = \int_s^t T(t-\sigma)\varphi(\sigma)\,d\sigma = \int_0^{t-s} T(\sigma)\varphi(t-\sigma)\,d\sigma \tag{29.3}$$

$$(0 \leqslant s < t < \infty)$$

is also continuously differentiable with respect to t, and the following representation exists:

$$\frac{dg}{dt} = T(t-s)\varphi(s) + \int_s^t T(t-\sigma)\varphi'(\sigma)\,d\sigma. \tag{29.4}$$

In addition to this, $\int_s^t T(t-\sigma)\varphi(\sigma)\,d\sigma \in D(A)$ and the following equation

$$\frac{dg}{dt} = \varphi(t) + A\int_s^t T(t-\sigma)\varphi(\sigma)\,d\sigma. \tag{29.4'}$$

is also valid.

Proof. Since $\|T(\tau)\| \leq 1$, the element $T(t-\sigma)\varphi(\sigma)$ for $\sigma\in[0,t]$ is continuous with respect to σ, as soon as function $\varphi(\sigma)$ becomes continuous. Therefore the integral

If we let

$$P_0 = I - \sum_{k=1}^{n} P_k, \tag{28.6}$$

the properties enumerated above can be extended to the operator P_0. (In accord- /182

ance with the theory of operational calculus for infinite closed operators (ref. 102), the projector P_0 may also be determined by means of the following equation

$$P_0 = I - \frac{1}{2\pi i} \int_{\Gamma_0} R(\lambda; A)\, d\lambda,$$

where Γ_0 is the envelope of the spectral set σ_0.)

In this chapter we shall show that in accordance with the spectrum structure of operator A, differential equation (28.1) may be split asymptotically into n+1 equations in the same manner as discussed in Chapter 3.

In this case each of the split equations is a differential equation in the corresponding subspace $P_k[\mathfrak{X}]$ and thus appears to be an equation of a "lower order" compared with the initial equation (28.1).

We shall begin the discussion of this problem by proving that the solution of equation (28.1), which satisfies condition (28.2) within the framework of propositions (S_1) and (S_2) indeed exists.

The next section will be devoted to this question.

Section 29. The Existence and Uniqueness of the Solution

In this section we shall not be concerned with the asymptotic properties of the solutions for equation (28.1) but will investigate only the question of its solvability. Therefore we rewrite (28.1) in the form

$$\frac{dx}{dt} = [A + B(t)]\, x, \qquad t \in \left[0, \frac{L}{\varepsilon}\right]. \tag{29.1}$$

The operators A and B(t) satisfy conditions (S_1) and (S_2)* respectively.

*To provide for the solvability of equation (29.1), condition (S_2) may be relaxed and replaced by the requirement that the function B(t)x be continuously differentiable with respect to t. Here and elsewhere in Section 29, unless a statement is made to the contrary, the terms continuity, differentiability and convergence are understood in the sense of a strong topology.

In the future we shall assume that the unbounded operator A satisfies conditions (S_1):

1) the region for determining D(A) of operator A is dense in $\mathfrak{X}$;
2) A is closed and linear; and
3) the resolvent of operator A exists for all $\lambda>0$ and satisfies the condition

$$\| R(\lambda; A)\| \equiv \| [A-\lambda I]^{-1}\| \leqslant \frac{1}{\lambda}, \qquad \lambda > 0.$$

The bounded linear operator $B(\tau, \varepsilon)$ is subjected to condition (S_2):

The operator-marking function $B(\tau, \varepsilon)$ is continuously differentiable with respect to $\tau\in[0, L]$ in the sense of the operators' norm.

We also assume that the spectrum $\sigma(A)$ of operator A is partitioned into n+1 spectral sets:

$$\sigma(A) = \bigcup_0^n \sigma_k, \tag{28.3}$$

where only the set σ_0 may contain the infinitely removed point while the remaining sets σ_k(k=1, 2, ..., n) are bounded.

The spectral set of the operator A, as we know (ref. 102), is the inaccessible set σ_k, which is the subset of spectrum $\sigma(A)$, which is simultaneously closed and open in $\sigma(A)$.

Let us assume further that Γ_k(k=1, 2, ..., n) is a closed smooth arc which surrounds the bounded spectral set σ_k and separates it from $\sigma(A)-\sigma_k$. According to the proposition, Γ_k lies in the resolvent set of operator A and bounds a certain open set Δ_k which contains σ_k.

The orientation of Γ_k is determined in the usual manner, i.e., when we move around Γ_k in the positive direction the set Δ_k remains on the left.

We introduce the projection operators for parallel projection P_k(k=1, 2, ..., n):

$$P_k = -\frac{1}{2\pi i}\int_{\Gamma_k} R(\lambda; A)\, d\lambda, \tag{28.4}$$

which, as we know (ref. 102), have the following properties:

$$P_k^2 = P_k; \qquad P_j P_k = 0 \ (j \neq k); \quad P_k D \subseteq D; \qquad P_k A \subseteq A P_k. \tag{28.5}$$

CHAPTER 5. THE ASYMPTOTIC SOLUTIONS OF DIFFERENTIAL EQUATIONS IN BANACH SPACE

Section 28. Formulation of the Problem

In the third chapter of this monograph we considered the question of the asymptotic partitioning of a system of linear differential equations of the n-th order into several independent systems of lower orders. The method which has been described can be generalized for the case of differential equations in spaces of infinite dimensions (Gilbert spaces, Banach spaces) as carried out by Yu. L. Baletskiy and S. G. Kreyn in a series of works (refs. 18-22). /180

In this chapter we shall present the basic results of these works, although for the convenience of the reader who is becoming acquainted with these problems for the first time, we shall limit ourselves to a case which is not the most general one. Those who are interested in a more general formulation of the problem are referred to references 20 and 22.

Thus, we shall consider the following differential equation in Banach space $\mathfrak{X}$:

$$\frac{dx}{dt} = [A + \varepsilon B(\tau, \varepsilon)]\, x, \tag{28.1}$$

in which $x(t,\varepsilon)$ is the unknown function of the variable $\tau \in \left[0, \frac{L}{\varepsilon}\right]$ with values in the space $\mathfrak{X}$; $A, B(\tau,\varepsilon)$ are operators which function in $\mathfrak{X}$; A is unbounded; $B(\tau,\varepsilon)$ is bounded (the parameters τ, ε and $L>0$ have the same character as before).

The solution of equation (28.1) which satisfies the given condition

$$x(t_0, \varepsilon) = x_0 \qquad (t_0 = 0), \tag{28.2}$$

where x_0 is some element from $\mathfrak{X}$ will be called the vector function $x(t,\varepsilon)$ possessing the following properties:

1) $x(t,\varepsilon)$ is highly continuous and continuously differentiable in any finite interval contained within $\left[0, \frac{L}{\varepsilon}\right]$;

2) $x(t,\varepsilon)$ for any $t>0$ belongs to the domain $D(A)$ of operator A and

$$\frac{dx}{dt} = [A + \varepsilon B(\tau, \varepsilon)]\, x;$$

3) $\lim\limits_{t \to 0+} \| x(t,\varepsilon) - x_0 \| = 0$

(For terminology and fundamental concepts of functional analysis see ref. 102.) /181

$$\times \left[1 + \frac{1}{4}\varepsilon^2 a(T)\right] \sin\alpha \cos\beta \cos K_2 + \varepsilon \left[1 + \frac{1}{4}\varepsilon^2 a(0)\right] \times$$

$$\times \left[1 - \frac{1}{4}\varepsilon^2 a(T)\right] \cos\alpha \sin\beta \cos K_2 = 0,$$

which we shall use to determine the eigenvalues of the boundary value problem (27.1)-(27.2).

Let us consider the following cases:

1) $\cos\alpha \neq 0, \qquad \cos\beta \neq 0.$

From equation (27.32) we have

$$\sin K_2 = 0,$$

from which we obtain the following asymptotic formula:

$$\lambda_n \approx \frac{\pi^2 n^2}{T^2},$$

or

$$\lambda_n \approx \frac{\pi^2 n^2}{T^2} - \frac{1}{T}\tan\alpha + \frac{1}{T}\tan\beta - \frac{1}{2T}\int_0^T a(\tau)\,d\tau;$$

2)

$$\lambda_n \approx \frac{(2n+1)^2}{4T^2}\pi^2 - \frac{1}{T}\tan\alpha - \frac{1}{2T}\int_0^T a(\tau)\,d\tau;$$

3)

$$\lambda_n \approx \frac{(2n+1)^2}{4T^2}\pi^2 + \frac{1}{T}\tan\beta - \frac{1}{2T}\int_0^T a(\tau)\,d\tau;$$

4)

$$\lambda_n \approx \frac{\pi^2 n^2}{T^2} \qquad (n = 1, 2, \ldots).$$

$$+ c_4 \{i\varepsilon^2 [(1 - \varepsilon^2 (a(0) + c'(0))) \sin\alpha - c(0)\cos\alpha](1 - e^{ik_2}) +$$
$$+ K\left[\varepsilon \sin\alpha \left(1 - \frac{1}{4}\varepsilon^2 a(0)\right) - i\cos\alpha \left(1 + \frac{1}{4}\varepsilon^2 a(0)\right)\right]\} = 0,$$

$$c_3 \left\{-i\varepsilon^2 [(1 - \varepsilon^2 (a(T) + c'(T))) \sin\beta - c(T)\cos\beta](1 - e^{ik_2}) e^{-\int_0^T c(\tau)d\tau} + \right.$$
$$\left. + K\left[\varepsilon \sin\beta \left(1 - \frac{1}{4}\varepsilon^2 a(T)\right) + i\cos\beta \left(1 + \frac{1}{4}\varepsilon^2 a(T)\right)\right] e^{ik_2}\right\} + \qquad (27.31)$$
$$+ c_4 \left\{i\varepsilon^2 [(1 - \varepsilon^2 (a(T) + c'(T))) \sin\beta - c(T)\cos\beta](1 - e^{-ik_2}) e^{-\int_0^T c(\tau)d\tau} + \right.$$
$$\left. + K\left[\varepsilon \sin\beta \left(1 - \frac{1}{4}\varepsilon^2 a(T)\right) - i\cos\beta \left(1 + \frac{1}{4}\varepsilon^2 a(T)\right)\right] e^{-ik_2}\right\} = 0.$$

In order that the system of equations (27.31) have a nontrivial solution, the determinant of the system must be equal to zero, i.e., the following relationship must be satisfied:

$$\varepsilon^2 \{[1 - \varepsilon^2 (a(0) + c'(0))] \sin\alpha - c(0)\cos\alpha\} \left[\varepsilon \left(\frac{1}{4}\varepsilon^2 a(T) - 1\right) \times\right.$$
$$\times \sin\beta \cos K_2 + \left(1 + \frac{1}{4}\varepsilon^2 a(T)\right) \cos\beta \sin K_2 +$$
$$\left. + \varepsilon \left(1 - \frac{1}{4}\varepsilon^2 a(T)\right) \sin\beta\right] + \varepsilon^2 \{[1 - \varepsilon^2 (a(T) + c'(T))] \sin\beta - c(T)\cos\beta\} \times$$
$$\times \left\{\varepsilon \left[\varepsilon^2 \frac{a(0)}{4} - 1\right] \sin\alpha \cos K_2 - \left[1 + \frac{1}{4}\varepsilon^2 a(0)\right] \cos\alpha \sin K_2 +\right.$$
$$\left. + \varepsilon \left[1 - \frac{1}{4}\varepsilon^2 a(0)\right] \sin\alpha\right\} e^{-\int_0^T c(\tau)d\tau} + K\left\{\varepsilon^2 \left[\frac{1}{4}\varepsilon^2 a(0) - 1\right] \times\right.$$
$$\times \left[1 - \frac{1}{4}\varepsilon^2 a(T)\right] \sin\alpha \sin\beta \sin K_2 - \left[1 + \frac{1}{4}\varepsilon^2 a(0)\right] \times$$
$$\left. \times \left[1 + \frac{1}{4}\varepsilon^2 a(T)\right] \cos\alpha \cos\beta \sin K_2 - \varepsilon \left[1 - \frac{1}{4}\varepsilon^2 a(0)\right] \times\right. \qquad (27.32)$$

From the boundary conditions (27.2) and taking into account (27.3), we obtain the following system for finding the constants c_1, c_2, c_3 and c_4:

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$$
\begin{aligned}
&c_1 - c_2 - c_3\varepsilon i + c_4\varepsilon i = 0,\\
&c_1\left[e^{-\int_0^T c(\tau)d\tau} + \int_0^T (1 + c(\tau))\, e^{-\int_0^\tau c(\tau)d\tau}\, d\tau\right] - c_2 - c_3\varepsilon i e^{ik_2} + c_4\varepsilon i e^{-ik_2} = 0,\\
&c_1\varepsilon\{[1 - \varepsilon^2(a(0) + c'(0))]\sin\alpha - c(0)\cos\alpha\} +\\
&+ c_3\left\{\left[1 - \frac{1}{4}\varepsilon^2 a(0)\right]\sin\alpha + i\left[1 + \frac{1}{4}\varepsilon^2 a(0)\right]\cos\alpha\right\} +\\
&+ c_4\left\{\left[1 - \frac{1}{4}\varepsilon^2 a(0)\right]\sin\alpha - i\left[1 + \frac{1}{4}\varepsilon^2 a(0)\right]\cos\alpha\right\} = 0,\\
&c_1\varepsilon\{[1 - \varepsilon^2(a(T) + c'(T))]\sin\beta - c(T)\cos\beta\}\, e^{-\int_0^T c(\tau)d\tau} +\\
&+ c_3\left\{\left[1 - \frac{1}{4}\varepsilon^2 a(T)\right]\sin\beta + i\left[1 + \frac{1}{4}\varepsilon^2 a(T)\right]\cos\beta\right\} e^{ik_2} +\\
&+ c_4\left\{\left[1 - \frac{1}{4}\varepsilon^2 a(T)\right]\sin\beta - i\left[1 + \frac{1}{4}\varepsilon^2 a(T)\cos\beta\right]\right\} e^{-ik_2} = 0,\\
&k_2 = \frac{T}{\varepsilon} + \frac{\varepsilon}{2}\int_0^T a(\tau)\, d\tau.
\end{aligned}
\tag{27.29}
$$

From the first two equations of (27.29) we have

$$
c_1 = \frac{i\varepsilon\,[c_4(1 - e^{-ik_2}) - c_3(1 - e^{ik_2})]}{K}, \tag{27.30}
$$

where

$$
K = e^{-\int_0^T c(\tau)d\tau} + \int_0^T [1 + c(\tau)] e^{-\int_0^\tau c(\tau)d\tau}\, d\tau - 1.
$$

Substituting the value of c_1 into the remaining two equations, we obtain an equation for determining c_3 and c_4:

$$
\begin{aligned}
&c_3\{-i\varepsilon^2[1 - \varepsilon^2(a(0) + c'(0))\sin\alpha - c(0)\cos\alpha](1 - e^{ik_2}) +\\
&+ K\left\{\varepsilon\sin\alpha\left(1 - \frac{1}{4}\varepsilon^2 a(0)\right) + i\cos\alpha\left(1 + \frac{1}{4}\varepsilon^2 a(0)\right)\right\} +
\end{aligned}
$$

Taking into account (27.24), we have

$$h_{11} = c_1\left[e^{-\int_0^\tau c(\tau)d\tau} + \int_0^\tau [1 + c(\tau)]\, e^{-\int_0^\tau c(\tau)d\tau}\right] - c_2,$$
$$h_{12} = -c_1\int_0^\tau [1 + c(\tau)]\, e^{-\int_0^\tau c(\tau)d\tau}\, d\tau + c_2. \qquad (27.26)$$

System (27.25) has the solution

$$h_{21} = c_3 \exp\left\{i\left[\frac{1}{\varepsilon}\tau + \frac{1}{2}\varepsilon\int_0^\tau a(\tau)\, d\tau\right]\right\},$$
$$h_{22} = c_4 \exp\left\{-i\left[\frac{1}{\varepsilon}\tau + \frac{1}{2}\varepsilon\int_0^\tau a(\tau)d\tau\right]\right\}. \qquad (27.27)$$

Thus, the vector x assumes the form

$$x = \begin{vmatrix} c_1\left[e^{-\int_0^t c(\tau)d\tau} + \int_0^t (1+c(\tau))\, e^{-\int_2^t c(\tau)d\tau}\, d\tau\right] - c_2 - c_3\varepsilon i e^{ik_1} + c_4\varepsilon i e^{-ik_1} \\ c_1 \exp\left\{-\int_0^t c(\tau)\, d\tau\right\} \end{vmatrix} \qquad (27.28)$$

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$$\begin{vmatrix} c_1\{1-\varepsilon^2[a(\tau)+c'(\tau)]\}e^{-\int_0^\tau c(\tau)d\tau} + c_3\left[1-\varepsilon^2\frac{a(\tau)}{4}\right]e^{ik_1} + c_4\left[1-\varepsilon^2\frac{a(\tau)}{4}\right]e^{-ik_1} \\ -c_1\varepsilon c(\tau)e^{-\int_0^t c(\tau)d\tau} + c_3 i\left(1+\varepsilon^2\frac{a(\tau)}{4}\right)e^{ik_1} - c_4 i\left(1-\varepsilon^2\frac{a(\tau)}{4}\right)e^{-ik} \end{vmatrix},$$

where

$$k_1 = \frac{\tau}{\varepsilon} + \varepsilon\frac{1}{2}\int_0^\tau a(\tau)\, d\tau,$$

c_1, c_2, c_3 and c_4 are arbitrary constants.

then equation (27.21) will assume the form

$$\frac{dh_1}{d\tau} = P(\tau)h_1. \qquad (27.23)$$

We shall seek a transformation matrix $Q(\tau)$

$$h_1 = Q(\tau)z_1, \qquad (27.24)$$

such that the matrix

$$R(\tau) = Q^{-1}(\tau)P(\tau)Q(\tau) - Q^{-1}(\tau)Q'(\tau)$$

has the form

$$R(\tau) = \begin{bmatrix} r_{11}(\tau) & 0 \\ r_{21}(\tau) & r_{22}(\tau) \end{bmatrix}.$$

In this case, equation (27.23) will be transformed into the equation

$$\frac{dz}{dt} = R(\tau)z. \qquad (27.25)$$

The matrix $Q(\tau)$ has the form

$$Q(\tau) = \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix}.$$

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Since the following condition is satisfied:

$$a(\tau)(p_{11} - p_{22}) + a^2(\tau)p_{21} - p_{12} = 0$$

when a=-1, we have y=const. (see ref. 24).

Therefore, when y=-1 we have

$$Q(\tau) = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \qquad R(\tau) = \begin{bmatrix} -c(\tau) & 0 \\ -1 - c(\tau) & 0 \end{bmatrix}.$$

From system (27.25) we obtain

$$z_1 = c_1 e^{-\int_0^\tau c(\tau)d\tau},$$

$$z_2 = -c_1 \int_0^\tau [1 + c(\tau)]\, e^{-\int_0^\tau c(\tau)d\tau} d\tau + c_2.$$

$$U_{12}(\tau)=\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -[a(\tau)+c'(\tau)] & -[a(\tau)+c'(\tau)] \\ 0 & 0 \end{bmatrix},$$

$$U_{22}(\tau)=\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -\frac{1}{4}a(\tau) & -\frac{1}{4}a(\tau) \\ \frac{i}{4}a(\tau) & -\frac{i}{4}a(\tau) \end{bmatrix}. \qquad (27.19)$$

Taking into account the established approximation for the matrices $\mathfrak{U}_k(\tau)$, $U_k(\tau)$, we obtain

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$$x=\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1-\varepsilon^2[a(\tau)+\varepsilon c'(\tau)] & 1-\varepsilon^2[a(\tau)+\varepsilon c'(\tau)] \\ -\varepsilon c(\tau) & -\varepsilon c(\tau) \end{bmatrix} h_1 + \\ +\begin{bmatrix} -\varepsilon i & \varepsilon i \\ 0 & 0 \\ 1-\frac{\varepsilon^2 a(\tau)}{4} & 1-\frac{\varepsilon^2 a}{4} \\ i\left(1+\frac{\varepsilon^2 a}{4}\right) & -i\left(1+\frac{\varepsilon^2 a}{4}\right) \end{bmatrix} h_2; \qquad (27.20)$$

h_1 and h_2 are determined by systems

$$\frac{dh_1}{dt}=\varepsilon\begin{bmatrix} 1 & 1 \\ -1-c(\tau) & -1-c(\tau) \end{bmatrix} h_1, \qquad (27.21)$$

$$\frac{dh_2}{dt}=\begin{bmatrix} i\left[1+\varepsilon^2\frac{a(\tau)}{4}\right] & 0 \\ 0 & -i\left[1+\varepsilon^2\frac{a(\tau)}{4}\right] \end{bmatrix} h_2. \qquad (27.22)$$

We shall transform the system (27.21) slightly. We let

$$P(\tau)=\begin{bmatrix} 1 & 1 \\ -1-c(\tau) & -1-c(\tau) \end{bmatrix}.$$

while $Q_{11}(\tau)$ is an arbitrary matrix. Let us assume that it is equal to zero. We select the matrix $\tilde{\tilde{Q}}_{21}(\tau)$ in such a way that

$$\{Q_{21}(\tau)\}_{s_2p_2} = \begin{cases} 0 \text{ for } s_2 = p_2, \\ \dfrac{\{D_{21}\}_{s_2p_2}}{\lambda_{p_20}(\tau) - \lambda_{s_20}(\tau)}, \quad s_2 \neq p_2, \quad s_2,\ p_2 = 3,\ 4. \end{cases}$$

Consequently, by taking into account the form of matrix D_{21}, we have

$$\tilde{\tilde{Q}}_{21}(\tau) = 0.$$

Then

$$\{\mathfrak{A}_{21}(\tau)\}_{s_2p_2} = \begin{cases} \{D_{21}\}_{s_2p_2} & s_2 = p_2, \\ 0 & s_2 \neq p_2,\ s_2,\ p_2 = 3,\ 4 \end{cases}$$

and

$$\mathfrak{A}_{21}(\tau) = 0. \tag{27.16}$$

From the first equation (27.13), when k=2 and from the second equation when k=1, we have /173

$$\tilde{Q}_{21}(\tau) = i\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \qquad \tilde{\tilde{Q}}_{11}(\tau) = \frac{i}{2}c(\tau)\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Consequently,

$$U_{11}(\tau) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -c(\tau) & -c(\tau) \end{bmatrix} \qquad U_{21}(\tau) = \begin{bmatrix} -i & i \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{27.17}$$

3. It can be shown that

$$\mathfrak{A}_{12}(\tau) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \mathfrak{A}_{22}(\tau) = \begin{bmatrix} \frac{i}{2}a(\tau) & 0 \\ 0 & -\frac{i}{2}a(\tau) \end{bmatrix}, \tag{27.18}$$

$$U_{10}(\tau) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}, \qquad U_{20}(\tau) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ i & -i \end{bmatrix}. \tag{27.12}$$

2. We determine $\mathfrak{A}_{11}(\tau), \mathfrak{A}_{21}(\tau)$ and $U_{11}(\tau), U_{21}(\tau)$ from the equation

$$U_{k1}(\tau)\mathfrak{A}_{k0}(\tau) - A_0(\tau)U_{k1}(\tau) = A_{10}(\tau)U_{k0}(\tau) - U_{k0}(\tau)\mathfrak{A}_{k1}(\tau),$$

which may be written in the form

$$Q_{k1}(\tau)W_k(\tau) - W(\tau)Q_{k1}(\tau) = D_{k1}(\tau) - Q_{k0}(\tau)\mathfrak{A}_{k1}(\tau), \tag{27.13}$$

where

$$Q_{k1}(\tau) = T^{-1}(\tau)U_{k1}(\tau), \qquad D_{k1}(\tau) = T^{-1}(\tau)A_{10}(\tau)U_{k0}(\tau) \tag{27.14}$$
$$(k = 1, 2).$$

From the construction of the matrix $W(\tau)$, equation (27.13), may be split into the following:

$$\begin{aligned} \widehat{Q}_{k1}(\tau)W_k(\tau) - W_1(\tau)\widehat{Q}_{k1}(\tau) &= \widetilde{D}_{k1}(\tau) - \widetilde{Q}_{k0}(\tau)\mathfrak{A}_{k1}(\tau), \\ \widetilde{\widetilde{Q}}_{k1}(\tau)W_k(\tau) - W_2(\tau)\widetilde{\widetilde{Q}}_{k1}(\tau) &= \widetilde{\widetilde{D}}_{k1}(\tau) - \widetilde{\widetilde{Q}}_{k0}(\tau)\mathfrak{A}_{k1}(\tau) \end{aligned} \tag{27.15}$$
$$(k = 1, 2).$$

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Then

$$D_{11}(\tau) = \begin{bmatrix} 1 & 1 \\ -1 - c(\tau) & -1 - c(\tau) \\ \frac{1}{2}c(\tau) & \frac{1}{2}c(\tau) \\ \frac{1}{2}c(\tau) & \frac{1}{2}c(\tau) \end{bmatrix} \equiv \begin{bmatrix} \widetilde{D}_{11} \\ \widetilde{\widetilde{D}}_{11} \end{bmatrix},$$

$$D_{21}(\tau) = \begin{bmatrix} 1 & 1 \\ -1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \equiv \begin{bmatrix} \widetilde{D}_{21} \\ \widetilde{\widetilde{D}}_{21} \end{bmatrix}.$$

When k=1, we obtain the following from the first equation (27.15)

$$\mathfrak{A}_{11}(\tau) = \widetilde{D}_{11}(\tau) = \begin{bmatrix} 1 & 1 \\ -1 - c(\tau) & -1 - c(\tau) \end{bmatrix},$$

Equating the coefficients of h_1 and h_2 in both sides of the equality, we obtain

$$\varepsilon U_k'(\tau, \varepsilon) + U_k(\tau, \varepsilon)\mathfrak{A}_k(\tau, \varepsilon) = [A_0(\tau) + \varepsilon A_1(\tau, \varepsilon)]U_k(\tau, \varepsilon) \quad (k = 1, 2).$$

It is easy to see that we shall obtain the following equations for determining

$$U_{ks}(\tau) \quad \text{and} \quad \mathfrak{A}_{ks}(\tau) \quad (k = 1, 2;\ s = 0, 1, 2, \ldots).$$

1. $U_{k0}(\tau)\mathfrak{A}_{k0}(\tau) = A_0(\tau)U_{k0}(\tau)$

or

$$Q_{k0}(\tau)\mathfrak{A}_{k0}(\tau) = W(\tau)Q_{k0}(\tau) \qquad (k = 1, 2), \tag{27.9}$$

where

$$Q_{k0}(\tau) = T^{-1}(\tau)U_{k0}(\tau) \qquad (k = 1, 2). \tag{27.10}$$

Each of equations (27.9) is partitioned into two independent equations /171

$$\begin{aligned} \widetilde{Q}_{k0}\mathfrak{A}_{k0}(\tau) &= W_1(\tau)\widetilde{Q}_{k0}(\tau), \\ \widetilde{\widetilde{Q}}_{k0}\mathfrak{A}_{k0}(\tau) &= W_2(\tau)\widetilde{\widetilde{Q}}_{k0}(\tau), \end{aligned} \qquad Q_{k0} = \begin{bmatrix} \widetilde{Q}_{k0}(\tau) \\ \widetilde{\widetilde{Q}}_{k0}(\tau) \end{bmatrix}.$$

We select the matrices $\widetilde{Q}_{10}(\tau)$, $\widetilde{\widetilde{Q}}_{20}(\tau)$, which are at our disposal, in such a way that

$$\widetilde{Q}_{10}(\tau) = E, \qquad \widetilde{\widetilde{Q}}_{20} = E.$$

Then

$$\mathfrak{A}_{k0}(\tau) = W_k(\tau) \qquad (k = 1, 2),$$

i.e.,

$$\mathfrak{A}_{10}(\tau) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \mathfrak{A}_{20}(\tau) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \tag{27.11}$$

while

$$\widetilde{\widetilde{Q}}_{10}(\tau) = \widetilde{Q}_{20}(\tau) = 0.$$

From equality (27.10), we have

has the form

$$W(\tau) = \begin{bmatrix} W_1(\tau) & 0 \\ 0 & W_2(\tau) \end{bmatrix},$$

where

$$W_1(\tau) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad W_2(\tau) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

In this case, the matrices $T(\tau)$ and $T^{-1}(\tau)$ are as follows:

$$T(\tau) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & i & -i \end{bmatrix}, \qquad T^{-1}(\tau) = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & -1 & 1 & -i \\ 0 & -1 & 1 & i \end{bmatrix}.$$

Since the elementary dividers are simple, the solution of equation (27.1) is sought in the form

$$x = U_1(\tau, \varepsilon) h_1 + U_2(\tau, \varepsilon) h_2, \tag{27.6}$$

where $U_1(\tau, \varepsilon)$ and $U_2(\tau, \varepsilon)$ are matrices of the order (4.2), and h_1 and h_2 are two-dimensional vectors which are determined from the equalities

$$\frac{dh_k}{dt} = \mathfrak{A}_k(\tau, \varepsilon) h_k \qquad (k = 1, 2). \tag{27.7}$$

In this case we assume that $\mathfrak{A}_k(\tau, \varepsilon)$ and $U_k(\tau, \varepsilon)$ can be represented in the form of formal power series in ε and

$$\begin{aligned} \mathfrak{A}_k(\tau, \varepsilon) &= \sum_{s=0}^{\infty} \varepsilon^s \mathfrak{A}_{ks}(\tau), \\ U_k(\tau, \varepsilon) &= \sum_{s=0}^{\infty} \varepsilon^s \mathfrak{A}_{ks}(\tau). \end{aligned} \tag{27.8}$$

Substituting (27.6) into system (27.5) and taking into account equality (27.7), we have

$$\sum_{k=1}^{2} [\varepsilon U_k'(\tau, \varepsilon) + U_k(\tau, \varepsilon) \mathfrak{A}_k(\tau, \varepsilon)] h_k =$$

$$= \sum_{k=1}^{2} [A_0(\tau) + \varepsilon A_1(\tau, \varepsilon)] U_k(\tau, \varepsilon) h_k.$$

$$\varepsilon \frac{dx_4}{dy} = x_2 - [1 + \varepsilon^2 a(y)]x_3. \qquad \text{/169}$$

By changing the variable

$$y = \varepsilon t = \tau$$

we reduce system (27.4) to the form

$$\frac{dx}{dt} = [A_0(\tau) + \varepsilon A_1(\tau, \varepsilon)]\, x, \qquad (27.5)$$

where

$$A_0(\tau) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix},$$

$$A_1(\tau) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -[c(\tau)+\varepsilon^2 b(\tau)] & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\varepsilon a(\tau) & 0 \end{bmatrix}.$$

Then the roots of the characteristic equation (12.3) will be as follows:

$$\lambda_{10}(\tau) = \lambda_{20}(\tau) = 0, \qquad \lambda_{30}(\tau) = i, \qquad \lambda_{40}(\tau) = -i.$$

We shall seek the solution of problem (27.1) and (27.2) in the form of an expansion in powers of ε, using the asymptotic method presented above.

As follows from Sections 20-25, the form of this expansion depends substantially on whether the elementary dividers corresponding to the multiple number $\lambda_{10}=0$ are multiple or simple. (In the first case the expansion will be in fractional powers of ε while in the second it will be in whole powers of ε). As we know, elementary dividers are simple when and only when the rank of the matrix $A_0(\tau)-\lambda E$ is equal to n-k, where n is the dimension of the matrix and k is the multiplicity of the number λ_{10}. In our case it is easy to see that the elementary dividers are simple. Thus, the canonical form of the matrix $A_0(\tau)$:

$$W(\tau) = T^{-1}(\tau) A_0(\tau) T(\tau)$$

The resulting system of linear differential equations (26.8) is a particular case of the system with form (17.1') which was considered by us in the preceding sections. In this connection it follows from the form of matrix $A_0(\tau)$ /168

that the characteristic equation (12.3) which has been constructed for the matrix $A_0(\tau)$ in this case has zero as its multiple root. Therefore, in constructing the asymptotic solution of system (26.8), we can use the method presented in Sections 20-25.

Without constructing the asymptotic solution of system (26.8) in the general case, we shall illustrate the idea of the method by means of the following example which is considered in reference 35.

Section 27. Calculation of the Eigenvalues of the Boundary Value Problem for a Differential Equation of the Fourth Order Containing Two Self-Conjugate Expressions

We shall seek the eigenvalues of the boundary value problem consisting of the equation

$$\frac{d^4u}{dy^4}+\frac{d}{dy}\left[a(y)\frac{du}{dy}\right]+b(y)u=-\lambda\left[\frac{d^2u}{dy^2}+C(y)u\right] \tag{27.1}$$

and boundary conditions

$$\begin{gathered}u(0)=0,\ u(T)=0,\\ \sin\alpha\left.\frac{du}{dy}\right|_{y=0}+\cos\alpha\left.\frac{d^2u}{dy^2}\right|_{y=0}=0,\\ \sin\beta\left.\frac{du}{dy}\right|_{y=T}+\cos\beta\left.\frac{d^2u}{dy^2}\right|_{y=T}=0.\end{gathered} \tag{27.2}$$

Using the substitutions

$$\begin{gathered}u=\varepsilon x_1,\qquad \frac{du}{dy}=\varepsilon x_3,\qquad \frac{d^2u}{dy^2}=x_4,\\ \varepsilon\frac{d^2u}{dy^2}+\varepsilon a(y)\frac{du}{dy}=x_2-x_3,\qquad \varepsilon=\frac{1}{\sqrt{\lambda}},\end{gathered} \tag{27.3}$$

we obtain a system

$$\begin{gathered}\frac{dx_1}{dy}=x_3,\\ \frac{dx_2}{dy}=-[c(y)+\varepsilon^2b(y)]x_1,\\ \varepsilon\frac{dx_3}{dy}=x_4,\end{gathered} \tag{27.4}$$

$$y = x_1, \quad \frac{dy}{d\tau} = x_2, \ldots, \frac{d^k y}{d\tau^k} = x_{k+1},$$

$$\varepsilon \frac{d^{k+1} y}{d\tau^{k+1}} = x_{k+2}, \ldots, \varepsilon^{l-1} \frac{d^{k+l-1} y}{d\tau^{k+l-1}} = x_{k+l}, \tag{26.4}$$

we shall reduce equation (26.1) to system (26.2), in which /167

$$A(\tau) = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ -a_0(\tau) & -a_1(\tau) & -a_2(\tau) & \ldots & -a_{k+l-1}(\tau) \end{bmatrix},$$

$$B(\tau) = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \varepsilon \dfrac{b(\tau)}{a_{k+l}(\tau)} \end{bmatrix}, \tag{26.5}$$

where

$$a_i(\tau) = \frac{\alpha_i(\tau)}{\alpha_{k+l}(\tau)} \quad (i = 0, 1, 2, \ldots, k+l-1). \tag{26.6}$$

Therefore, in the future, in place of equation (26.1) we shall consider the equivalent system (26.2). According to references 95 and 96, the latter system can be transformed by means of the substitution

$$\tau = \varepsilon t \tag{26.7}$$

into the system

$$\frac{dx}{dt} = [A_0(\tau) + \varepsilon A_1(\tau)]\, x + \varepsilon B(\tau) e^{i\theta(t,\varepsilon)}, \tag{26.8}$$

where

$$A_0(\tau) = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 \\ a_{k+1,1}(\tau) & a_{k+1,2}(\tau) & \ldots & a_{k+1,n}(\tau) \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1}(\tau) & a_{n2}(\tau) & & a_{nn}(\tau) \end{bmatrix},$$

$$A_1(\tau) = \begin{bmatrix} a_{11}(\tau) & a_{12}(\tau) & \ldots & a_{1n}(\tau) \\ \ldots & \ldots & \ldots & \ldots \\ a_{k1}(\tau) & a_{k2}(\tau) & \ldots & a_{kn}(\tau) \\ 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 \end{bmatrix}, \tag{26.9}$$

$a_{jj}(\tau)$ (i, j=1, 2, ..., n) are elements of the matrix A(τ).

Section 26. Differential Equations With Small Parameter in Front of the Higher-Order Derivatives

As we mentioned in the introduction, the asymptotic method which we have applied to differential equations with slowly varying coefficients may also be applied to the differential equations frequently investigated in references 78-80, 16-17 and 9-12, which contain a small parameter as a multiplier in front of the higher-order derivatives. A peculiarity of the characteristic equation in this case is that is has zero as its multiple root. /166

Thus, differential equations with a small parameter in front of the higher-order derivatives may serve as an example of differential equations considered by us in Sections 20-25 of this chapter.

Thus let us consider the differential equation

$$\sum_{i=0}^{k} a_i(\tau)\frac{d^i y}{d\tau^i} + \sum_{r=1}^{l} \varepsilon^r a_{k+r}(\tau)\frac{d^{k+l} y}{d\tau^{k+l}} = \varepsilon b(\tau) e^{i\theta(\tau)}, \tag{26.1}$$

in which $a_h(\tau)$ $(h=1, 2, \ldots, k+l)$ are functions on the segment $0 \leq \tau \leq L$, $(a_{k+l}(\tau) \neq 0, \tau \in [0, L])$, ε which can be differentiated a sufficient number of times.

Equation (26.1) may be looked upon as a particular case of a system of linear differential equations

$$E_1 \frac{dx}{d\tau} = A(\tau)x + E_1 B(\tau) e^{i\theta(\tau)}, \tag{26.2}$$

where x, $B(\tau)$ are n-dimensional vectors, $A(\tau)$ is a real square matrix of order n, and E_1 is a diagonal matrix of the following form:

$$E_1 = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ & & & & \varepsilon \\ & & & & & \ddots \\ 0 & & & & & & \varepsilon \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}1\\&\ddots\\&&1\end{matrix}}\right\} k \\ \\ \left.\vphantom{\begin{matrix}\varepsilon\\&\ddots\\&&\varepsilon\end{matrix}}\right\} n-k \end{matrix} \tag{26.3}$$

Indeed, if we let

where

$$W_0(\tau)=\begin{bmatrix}\lambda_{10}(\tau) & 1\\ 0 & \lambda_{10}(\tau)\end{bmatrix},\qquad W_1(\tau)=\begin{bmatrix}\{C(\tau)\}_{11} & \{C(\tau)\}_{12}\\ 0 & \{C(\tau)\}_{22}\end{bmatrix}, \tag{25.80}$$

(in view of (25.1), $\{C(\tau)\}_{21}=0$).

We subject system (25.78) to the following transformation:

$$\xi_1=q_1,\qquad \xi_2=\varepsilon q_2. \tag{25.81}$$

Then system (25.78) will assume the form

$$\frac{dq}{dt}=[\lambda_{10}(\tau)E+\varepsilon\widetilde{W}(\tau,\varepsilon)]\,q, \tag{25.82}$$

where

$$\widetilde{W}(\tau,\varepsilon)=\begin{bmatrix}\{W(\tau,\varepsilon)\}_{11} & 1+\varepsilon\{W(\tau,\varepsilon)\}_{12}\\ \{W(\tau,\varepsilon)\}_{21} & \{W(\tau,\varepsilon)\}_{22}\end{bmatrix}, \tag{25.83}$$

q is a two dimensional vector; $q=\left[\frac{q_1}{q_2}\right]$.

By applying the exponential transformation

$$q=\exp\left\{\int_0^t\lambda_{10}(\tau)\,dt\right\}v, \tag{25.84}$$

to the system (25.82), we obtain

$$\frac{dv}{dt}=\varepsilon\widetilde{W}(\tau,\varepsilon)v,$$

or

$$\frac{dv}{d\tau}=\widetilde{W}(\tau,\varepsilon)v. \tag{25.85}$$

The resulting system (25.85). unlike system (17.4), has a zero rank (see reference 76, page 80).

Therefore, the method of successive approximations can be applied to it. It follows that we can obtain a solution in the form of a converging series in powers of ε, for the vector v and consequently for ξ.

(according to expansion (25.69), $\widetilde{P}_0(\tau)=0$). Since

$$\det [A_0(\tau) - i\nu(\tau) E] \neq 0 \tag{25.73}$$

for any $\tau\in[0,L]$, we find the following expression from equality (25.71):

$$\widetilde{P}_s(\tau) = [A_0(\tau) - i\nu(\tau) E]^{-1}\Phi_s(\tau) \qquad (s = 1, 2, \ldots). \tag{25.74}$$

3. By applying the method which is described in Section 23, we could show that the formal solutions (25.4) and (25.67) which have been constructed here have an asymptotic nature and that, specifically, the following theorem is valid.

Theorem 4.9. If the conditions of Theorem 4.7 are satisfied for all $\tau\in[0,L]$

$$\operatorname{Re}\left(\sum_{s=0}^{k_j-2}\mu_j^s\lambda_{js}(\tau)\right) < 0, \tag{25.75}$$

then for any $L>0$ and $0<\varepsilon\leq\varepsilon_0$ we can specify a constant C independent of ε such that

$$\|x - x^{(m)}\| \leqslant \mu_1^{m+2-2k_1}C \tag{25.76}$$

(It is assumed that $x|_{t=0} = x^{(m)}|_{t=0}$ and $k_1 \geqslant k_2 \geqslant \ldots \geqslant k_p$).

In the case of "nonresonance" we can obtain the estimate

$$\|x - x^{(m)}\| \lesssim \mu_1^{m+2-k_1}C. \tag{25.77}$$

Note. The algorithm proposed by us for constructing the asymptotic solution of system (17.1') when we have doubly multiple roots of the characteristic equations cannot be effective, because in this case it leads to the Riccati equation. Therefore, in this case it is expedient to apply the method presented below.

Following Section 13, we can asymptotically partition system (17.1') into p subsystems of order k_j ($j=1, 2, \ldots, p$). For example, let $k_1=2$. Then the corresponding separated system (for simplicity we consider the case of "nonresonance") has the form

$$\frac{d\xi}{dt} = W(\tau, \varepsilon)\xi, \tag{25.78}$$

in which ξ is a two-dimensional vector $\xi[\frac{\xi_1}{\xi_2}]$, while $W(\tau,\varepsilon)$ is a square matrix of the second order /165

$$W(\tau, \varepsilon) = \sum_{s=0}^{m}\varepsilon^s W_s(\tau), \tag{25.79}$$

2. A similar algorithm for constructing the formal solution of system (17.1') may also be proposed in the case of "nonresonance." Specifically, this proposition has the form of the following theorem.

Theorem 4.8. If the conditions of 4.7 are satisfied, the formal general solution of system (17.1') in the case of "nonresonance" may be represented in the form

$$x = \sum_{i=1}^{p} U_i(\tau, \mu_i)\tilde{h}_i + \tilde{P}(\tau, \varepsilon)e^{i\theta(t,\varepsilon)}, \tag{25.67}$$

$$\frac{d\tilde{h}_i}{dt} = \lambda_i(\tau, \mu_i)\tilde{h}_i \qquad (j = 1, 2, \ldots, p), \tag{25.68}$$

where

$$U_j(\tau, \mu_j), \qquad \lambda_j(\tau, \mu_j) \qquad (j = 1, 2, \ldots, p)$$

are the same as in Theorem 4.7, while $\tilde{P}(\tau, \varepsilon)$ is an n-dimensional vector which admits a formal expansion

$$\tilde{P}(\tau, \varepsilon) = \sum_{s=1}^{\infty} \varepsilon^s \tilde{P}_s(\tau). \tag{25.69}$$

Proof. Since in subsection (1) of the present section we specified a method for determining the vectors $U_j(\tau, \mu_j)$ and the functions $\lambda_j(\tau, \mu_j)$ (j=1, 2, ..., p), then to prove the given theorem it is necessary to specify the method of determining the coefficients of vector $\tilde{P}(\tau, \varepsilon)$.

If the vector x, which is determined by relationships (25.67) and (25.68), is substituted into the system we shall obtain an identity. If the coefficients in front of $\tilde{h}_j$ and the free terms are equated, we obtain relationship (25.9) and relationship

$$[A(\tau, \varepsilon) - i\nu(\tau)E]\tilde{P}(\tau, \varepsilon) = \varepsilon(\tilde{P}'(\tau, \varepsilon) - b(\tau, \varepsilon)). \tag{25.70}$$

By isolating coefficients of order ε^s (s=1, 2, ...), we have

$$[A_0(\tau) - i\nu(\tau)E]\tilde{P}_s(\tau) = \Phi_s(\tau), \tag{25.71}$$

where

$$\Phi_s(\tau) = \tilde{P}'_{s-1}(\tau) - b_{s-1}(\tau) - \sum_{i=1}^{s} A_i(\tau)\tilde{P}_{s-1}(\tau) \tag{25.72}$$ /164

has a unique solution which may be written in the form

$$x(t) = V(t, t_0)\, x_0, \quad x_0 \in D(A).$$

Let us now prove the second part of Theorem 5.1, specifically that the function $V(t,s)x_0$ ($x_0 \in D(A)$) is continuously differentiable with respect to the parameter s ($0 \le s \le t < \infty$).

Indeed, the function $V_0(t,s,)x_0 = W_0(t,s)x_0 = T(t-s)x_0$ is continuously differentiable with respect to $s \in [0,t]$ when $x_0 \in D(A)$. In this case

$$\frac{dW_0(t,s)}{ds} x_0 = -T(t-s)Ax_0 = -W_0(t,s)Ax_0. \tag{29.20}$$

Consequently, in view of the supplement to Lemma 5.1 the derivative $\frac{dW_1(t,s)}{ds} x_0$ of the function $W_1(t,s)$ (see (29.7)) exists and is continuous with respect to t and s:

$$\frac{dW_1(t,s)}{ds} x_0 = -T(t-s)B(s)\, x_0 - \int_s^t T(t-\sigma)B(\sigma)W_0(\sigma,s)Ax_0 d\sigma =$$

$$= -W_0(t,s)B(s)x_0 - W_1(t,s)Ax_0.$$

By induction we find

$$\frac{dW_n(t,s)}{ds} x_0 = -W_{n-1}(t,s)B(s)x_0 - W_n(t,s)Ax_0, \tag{29.21}$$

and $\frac{d}{ds} W_n(t,s)x_0$ is continuous with respect to t and s ($0 \le s \le t < \infty$).

From equations (29.20) and (29.21) we obtain estimates which are analogous to inequalities (29.12):

$$\left\| \frac{dW_0(t,s)}{ds} x_0 \right\| \le \| Ax_0 \|,$$

$$\left\| \frac{dW_n(t,s)}{ds} x_0 \right\| \le M^n \frac{(t-s)^{n-1} + (t-s)^n}{(n-1)!} \alpha \quad (n = 1, 2, \ldots).$$

According to these estimates we are able to maintain that the function

$$x(t,s) = V(t,s)x_0 = \sum_{n=0}^{\infty} W_n(t,s)x_0 \quad (x_0 \in D(A))$$

is continuously differentiable with respect to the parameter s, while the derivative $\frac{dV(t,s)}{ds} x_0$ is continuous with respect to variables t and s. For the derivative $\frac{dV(t,s)}{ds} x_0$ we have the following estimate

$$\left\| \frac{dV(t,s)}{ds} x_0 \right\| \leqslant [1 + Me^{M(t-s)}(1 + t - s)]\,\alpha, \tag{29.22}$$

where

$$\alpha = \| Ax_0 \| + \| x_0 \|.$$

In addition to this, in view of (29.20) and (29.21) the following relationship exists

$$\frac{dV(t,s)}{ds} x_0 = -V(t,s)B(s)x_0 - V(t,s)Ax_0 = -V(t,s)[A + B(s)]x_0.$$

This concludes the proof of Theorem 5.1.

Now let us consider the construction of the solution to the nonhomogeneous equation

$$\frac{dx}{dt} = [A + B(t)]x + f(t) \tag{29.23}$$

with initial conditions

$$x(t_0) = x_0 \in D(A)\ (0 \leqslant t_0 \leqslant t < \infty). \tag{29.24}$$

Here we have the following theorem.

Theorem 5.2. Let the operators A and B(t) be the same as in Theorem 5.1, and let the function $f(t) \in \mathfrak{X}$ be continuously differentiable.

Then a unique solution of problem (29.23), (29.24) exists. This solution has the form

$$x(t) = V(t,t_0)x_0 + \int_{t_0}^{t} V(t,s)f(s)\,ds,$$

where V(t,s) is the operator from Theorem 5.1.

Proof. Let us construct the sequence

$$y_0(t) = \int_{t_0}^{t} W_0(t, s) f(s)\, ds = \int_{t_0}^{t} T(t-s) f(s)\, ds,$$
$$y_n(t) = \int_{t_0}^{t} W_n(t, s) f(s)\, ds \quad (n = 1, 2, \ldots), \tag{29.25}$$

where $W_n(t,s)$ are the same operators as in Theorem 5.1.

It is easy to see that

$$y_n(t) = \int_{t_0}^{t} T(t-\sigma) B(\sigma) y_{n-1}(\sigma)\, d\sigma \quad (n = 1, 2, \ldots). \tag{29.26}$$

Indeed /192

$$y_n(t) = \int_{t_0}^{t} \left[\int_{s}^{t} T(t-\sigma) B(\sigma) W_{n-1}(\sigma, s)\, d\sigma \right] f(s)\, ds =$$
$$= \int_{t_0}^{t} T(t-\sigma) B(\sigma) \left[\int_{t_0}^{\sigma} W_{n-1}(\sigma, s) f(s)\, ds \right] d\sigma =$$
$$= \int_{t_0}^{t} T(t-\sigma) B(\sigma) y_{n-1}(\sigma)\, d\sigma.$$

According to Lemma 5.1 the function $y_0(t)$ is continuously differentiable.

y using the method of induction we can see that this is also true for any $y_n(t)$, nd that

$$y_0'(t) = T(t-t_0) f(t_0) + \int_{t_0}^{t} T(t-s) f'(s)\, ds,$$
$$y_n'(t) = \int_{t_0}^{t} T(t-s) B'(s) y_{n-1}(s)\, ds + \int_{t_0}^{t} T(t-s) B(s) y_{n-1}'(s)\, ds \tag{29.27}$$
$$(n = 1, 2, \ldots).$$

By using expressions (29.26) and (29.27), it is easy to obtain the follow- g estimates

$$\| y_n(t) \| \leqslant M^n K \frac{(t-t_0)^{n+1}}{(n+1)!},$$
$$\| y_n'(t) \| \leqslant M^n K \frac{(t-t_0)^n + (t-t_0)^{n+1}}{n!}, \tag{29.28}$$

where

$$K = \max_{0 \leqslant t_0 \leqslant t < \infty} [\| f(t) \|, \ \| f'(t) \|].$$

Assuming that

$$y(t) = \sum_{n=0}^{\infty} y_n(t), \tag{29.29}$$

and in accordance with inequalities (29.28) we can maintain that the series (29.29) converges uniformly with respect to t and can be differentiated term-by-term, and that y(t) is a continuously differentiable function.

According to equations (29.25) and (29.26), we have the following representations for the function y(t) which has been obtained:

$$y(t) = \int_{t_0}^{t} T(t-s) f(s)\, ds + \int_{t_0}^{t} T(t-s) B(s) y(s)\, ds = \tag{29.30}$$

$$= \int_{t_0}^{t} V(t,s) f(s)\, ds,$$

and $y(t_0)=0$.

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From Lemma 5.1 and from expression (29.30) it follows that

$$\frac{dy}{dt} = f(t) + A \int_{t_0}^{t} T(t-s) f(s)\, ds + B(t) y(t) +$$

$$+ A \int_{t_0}^{t} T(t-s) B(s) y(s)\, ds = f(t) + [A + B(t)] y(t),$$

i.e., the function $y(t) = \int_{t_0}^{t} V(t,s) f(s)\, ds$ is a particular solution of equation (29.23).

Thus by combining the results of Theorems 5.1 and 5.2 we come to the conclusion that the vector

$$x(t) = V(t, t_0) x_0 + \int_{t_0}^{t} V(t,s) f(s)\, ds$$

is a solution of problem (29.23) and (29.24), which was to be proved.

Note. Reference 70 shows that equation (29.20) can be solved under less rigid propositions concerning the coefficients and the initial conditions $x(0)=x_0$.

In particular the operator B(t) may also be unbounded but must satisfy the operator A in a definite sense.

Section 30. The Solvability of Certain Operational Equations in a Banach Space

In the asymptotic partition of equation (28.1) we shall frequently encounter an equation of the form

$$AX - XA = F, \tag{30.1}$$

where A is our assigned operator which satisfies conditions (S_1); F is some bounded operator.

The functional equation (30.1) is interpreted in the sense that the left and right sides of the equation coincide on vectors ψ from domain D(A):

$$(AX - XA)\psi = F\psi \ (\psi \in D(A)).$$

Let α be some point in the complex plane λ which does not belong to the spectrum $\sigma(A)$. Then the operator $(A-\alpha I)^{-1}=H$ is a limited operator whose spectrum is obtained from the spectrum $\sigma(A)$ by means of the fractionally linear transformation /194

$$\mu = \frac{1}{\lambda - \alpha}. \tag{30.2}$$

(It is assumed that the selection of α provides for the fulfillment of the requirement $\mu \in \sigma(A)$.)

Thus if it is specified that the spectrum $\sigma(A)$ partitions into n+1 spectral sets $\sigma_k \left(\sigma(A)=\overset{n}{\underset{0}{U}}\sigma_k\right)$, then the spectrum of the operator H also consists of n+1 closed parts

$$\sigma(H) = \overset{n}{\underset{0}{U}} \sigma'_k,$$

and $\sigma'_k = \frac{1}{\sigma_k - \alpha}$ $(k = 0, 1, 2, \ldots, n)$ are bounded sets and only σ'_0 contains the point $\mu=0$.

Furthermore if Γ_k is an arc which separates the spectral set σ_k from the remaining part of the spectrum $\sigma(A)-\sigma_k$ while $\widetilde{\Gamma}_k$ contains $\sigma(A)-\sigma_k$, then as a result of transformation (30.2) we shall obtain an arc γ_k which surrounds the set σ'_k and the arc $\widetilde{\gamma}_k$ which contains $\sigma(H)-\sigma'_k$.

Let us construct the projection operator J_k for the operator H so that it corresponds to the spectral set σ_k':

$$J_k = -\frac{1}{2\pi i}\int_{\gamma_k} R(\mu; H)\,d\mu \quad (k = 0, 1, 2, \ldots, n). \tag{30.3}$$

The projectors J_k have the same properties of the operators P_k:

$$J_k^2 = J_k; \quad J_k J_j = 0 \ (k \neq j), \ J_k H = H J_k.$$

Also it is easy to show that $J_k \equiv P_k$ (k=0, 1, ..., n).

To do this we first represent the resolvent of operator H in terms of the resolvent of the assigned operator A. At this point it becomes desirable to use the well-known equation from generalized operational calculus (ref. 102):

$$f(A) = f(\infty)I - \frac{1}{2\pi i}\int_{\Gamma} f(\lambda) R(\lambda; A)\,d\lambda, \tag{30.4}$$

where $\Gamma = \bigcup_0^n \Gamma_k$, is the oriented envelope of the set $\Delta \supset \sigma(A)$; $f(\lambda)$ is some complex function which is holomorphic in Δ.

It is obvious that to suit our purpose of constructing the resolvent of operator H

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$$R(\mu; H) = [(A - \alpha I)^{-1} - \mu I]^{-1}, \tag{30.5}$$

we should select the function $f(\lambda)$ as follows:

$$f(\lambda) = [(\lambda - \alpha)^{-1} - \mu]^{-1}. \tag{30.6}$$

Then, according to equation (30.4), we find

$$R(\mu; H) = -\frac{1}{\mu} I - \frac{1}{2\pi i}\int_{\Gamma} \frac{R(\lambda; A)\,d\lambda}{\frac{1}{\lambda - \alpha} - \mu}$$

and consequently the operator J_k has the form

$$J_k = -\frac{1}{2\pi i}\int_{\gamma_k} R(\mu; H)\,d\mu = \frac{1}{2\pi i}\int_{\gamma_k} \frac{d\mu}{\mu} I +$$

$$+ \left(\frac{1}{2\pi i}\right)^2 \int_{\gamma_k}\int_{\Gamma} \frac{R(\lambda; A)}{\frac{1}{\lambda - \alpha} - \mu}\,d\lambda\,d\mu.$$

Using the conventional arguments from the theory of the functions of a complex variable, we can write out the following chain of equalities and carry out the appropriate proof:

$$J_k = \delta I + \left(\frac{1}{2\pi i}\right)^2 \int\limits_{\gamma_k}\int\limits_{\Gamma_k} \frac{R(\lambda;\ A)}{\frac{1}{\lambda-\alpha}-\mu}\,d\lambda d\mu +$$

$$+\left(\frac{1}{2\pi i}\right)^2 \int\limits_{\gamma_k}\int\limits_{\widetilde{\Gamma}_k} \frac{R(\lambda;\ A)}{\frac{1}{\lambda-\alpha}-\mu}\,d\lambda d\mu = \delta I + \left(\frac{1}{2\pi i}\right)^2 \int\limits_{\Gamma_k} R(\lambda;\ A)\times$$

$$\times\left[\int\limits_{\gamma_k} \frac{d\mu}{\frac{1}{\lambda-\alpha}-\mu}\right]d\lambda + \left(\frac{1}{2\pi i}\right)^2 \int\limits_{\widetilde{\Gamma}_k} R(\lambda;\ A)\left[\int\limits_{\gamma_k} \frac{d\mu}{\frac{1}{\lambda-\alpha}-\mu}\right]d\lambda =$$

$$= \delta I - \frac{1}{2\pi i}\int\limits_{\Gamma_k} R(\lambda;\ A)\,d\lambda = P_k,$$

where

$$\delta = \begin{cases} 1, & \text{if } k = 0. \\ 0, & \text{if } k \neq 0. \end{cases}$$

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Let us now consider the operational equation for

$$HX - XH = G, \tag{30.7}$$

where H is the constructed operator, while G is some (known) bounded operator.

It is easy to show that the following theorem is valid.

Theorem 5.3. Let us assume that H and G are assigned bounded operators in Banach space $\mathfrak{X}$.

When the conditions

$$GP_k = G,\ P_kG = 0 \tag{30.8}$$

are satisfied equation (30.7) has the solution

$$X = -\frac{1}{2\pi i}\int\limits_{\gamma_k} R(\mu;\ H)\,GR(\mu;\ H)\,d\mu. \tag{30.9}$$

$$XP_k = X; \qquad P_kX = 0. \tag{30.10}$$

The proof of Theorem 5.3 will involve a direct verification of solution (30.9).

In this case we should remember that according to the proof, $J_k=P_k$.

Let us substitute expression (30.9) into both sides of equation (30.7). As a result we obtain

$$HX = -\frac{1}{2\pi i}\int_{\gamma_k} GR(\mu; H)\,d\mu - \frac{1}{2\pi i}\int_{\gamma_k} \mu R(\mu; H)GR(\mu; H)\,d\mu,$$
$$XH = -\frac{1}{2\pi i}\int_{\gamma_k} R(\mu; H)\,Gd\mu - \frac{1}{2\pi i}\int_{\gamma_k} \mu R(\mu; H)\,GR(\mu; H)d\mu. \qquad (30.11)$$

Here we have used the well-known property of the resolvent for bounded operators:

$$HR(\mu; H) = R(\mu; H)H = I + \mu R(\mu; H);\quad \mu \overline{\in} \sigma(H).$$

It follows from equalities (30.11) that

$$HX - XH = GP_k - P_kG = G,$$

which is confirmed by the theorem.

It is easy to show directly that solution (30.9) satisfies condition (30.10). We shall now show that a solution with these properties is unique.

Let us assume that along with solution X there is one other solution Y /197 which satisfies condition (30.10), i.e.,

$$YP_k = Y,\ P_kY = 0.$$

Then obviously the element Z=X-Y is a solution of the homogeneous equation

$$HZ - ZH = 0. \qquad (30.12)$$

Since the right side of equation (30.12) satisfies condition (28.8) (in this case G≡0), we can use the above proof to construct an element Z by means of equation (30.9). As a result of this we obtain Z≡0, i.e., X≡Y, which was to be proved.

We note in conclusion that the solution of equation (30.7) which satisfies conditions (30.8) may also be written in the form

$$X = \frac{1}{2\pi i}\int_{\widetilde{\gamma}_k} R(\mu;\ H)\,GR(\mu;\ H)\,d\mu \qquad (30.9')$$

(compare with equation (30.9)).

Now we can return to the initial equation (30.1) and show that when the conditions

$$FP_k = F,\ P_kF = 0 \tag{30.13}$$

are satisfied, equation (30.9), in which

$$G = -HFH, \tag{30.14}$$

is a solution of equation (30.1) which has the property (30.10).

We note that in view of the equality $P_kH=HP_k$, which is valid for limited operators, the fulfillment of condition (30.13) also causes the fulfillment of the same conditions for the operator G from (30.14).

We establish first of all that the operator (30.9) leaves the domain D(A) invariant, i.e., $XD \subset D$.

Indeed, from equation (30.7), we have

$$XH = HX - G = [A - \alpha I]^{-1} X + [A - \alpha I]^{-1} F [A - \alpha I]^{-1} =$$
$$= [A - \alpha I]^{-1} [X + F(A - \alpha I)^{-1}].$$

Let us assume that $f \in D(A)$ and $(A-\alpha I)f=g$, i.e., $f=Hg$.

Then we can write

$$Xf = XHg = [A - \alpha I]^{-1} [X + FH] g, \tag{30.15}$$

from which it follows that $Xf \in D$, because the bounded operator $H=(A-\alpha I)^{-1}$ transforms the entire space $\mathfrak{X}$ into D(A). Let us apply the operator $(A-\alpha I)$ to the left and right sides of equality (30.15): /198

$$(A - \alpha I) Xf = Xg + FHg = X(A - \alpha I) f + Ff$$

As a result we obtain the required equality

$$AXf - XAf = Ff,\ f \in D(A).$$

Thus we have established the following theorem.

Theorem *5.4. Let the operator A be the same as in Theorem 5.1 and let the bounded operator F satisfy conditions (30.13).

Then the operational equation (30.1) has the solution

The solution of operational equations of a more general type is considered in reference 21.

$$X = \frac{1}{2\pi i}\int_{\gamma_k} R(\mu; H) HFHR(\mu; H)\, d\mu, \tag{30.16}$$

which is unique and which satisfies the condition

$$XP_k = X,\ P_kX = 0.$$

Note. Let us assume that the operator F depends on the parameter τ:

$$F \equiv F(\tau),\ \tau \in [0,\ L].$$

Then it follows from the form of equation (30.16) that the solution $x(\tau)$ will have the same number of derivatives with respect to τ as $F(\tau)$.

Later in this chapter differentiability, continuity and convergence are interpreted in the sense of the operators' norm.

Section 31. The Construction of the Formal Solution

Let us return to the equation

$$\frac{dx}{dt} = [A + \varepsilon B(\tau, \varepsilon)]\, x \quad \left(0 \leqslant t \leqslant \frac{L}{\varepsilon}\right) \tag{31.1}$$

and specify a method for constructing its formal solution, which satisfies the condition

$$x(t_0) = x_0,\ x_0 \in D(A),\ 0 \leqslant t_0 \leqslant t \leqslant \frac{L}{\varepsilon}. \tag{31.2}$$

In addition to the propositions which have already been formulated concerning the properties of operators A and $B(\tau, \varepsilon)$ (see Section 28), we shall also assume that the operator $B(\tau, \varepsilon)$ can be represented in the form

$$B(\tau, \varepsilon) = \sum_{s=1}^{\infty} \varepsilon^{s-1} B_s(\tau),\ \| B(\tau, \varepsilon) \| \leqslant M_1, \tag{31.3}$$

where $B_s(\tau)$ can be differentiated with respect to τ a sufficient (infinite) /199

number of times.

It follows from Theorem 5.1 that problem (31.1) and (31.2) has an exact solution of the form

$$x(t, \varepsilon) = V(t, t_0, \varepsilon) x_0, \quad x_0 \in D(A), \tag{31.4}$$

where $V(t, t_0, \varepsilon)$ is the operator investigated in Section 29 which satisfies the conditions:

$$V(t_0, t_0, \varepsilon) = I,$$
$$\| V(t, t_0, \varepsilon) \| \leqslant e^{\varepsilon M_1 (t - t_0)} \leqslant e^{M_1 L}.$$

Taking into account the spectrum structure of operator A (see (28.3)), we shall construct the operator $V(t, t_0, \varepsilon)$ for the formal solution of equation (31.1) in the form

$$V(t, t_0, \varepsilon) = \sum_{k=0}^{n} V_k(\tau, \varepsilon) X_k(t, t_0, \varepsilon), \tag{31.5}$$

assuming that the operators $X_k(t, t_0, \varepsilon)$ are effective in subspaces $\mathfrak{X}_k = P_k[\mathfrak{X}]$:

$$X_k(t, t_0, \varepsilon) = P_k X_k(t, t_0, \varepsilon) = X_k(t, t_0, \varepsilon) P_k \tag{31.6}$$

and satisfy the following equations in these subspaces:

$$\frac{dX_k}{dt} = \Omega_k(\tau, \varepsilon) X_k(t, t_0, \varepsilon) \qquad (k = 0, 1, 2, \ldots, n). \tag{31.7}$$

Equation (31.7) is interpreted as the equivalence of the left and right sides on vectors from domain D(A).)

It is assumed that the following representatives are valid for coefficients from expressions (31.5) and (31.7):

$$V_k(\tau, \varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s V_{ks}(\tau),$$
$$\Omega_k(\tau, \varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s \Omega_{ks}(\tau), \tag{31.8}$$

we say nothing about the nature of convergence of the above.

Before proceeding with the construction of operators $V_k(\tau, \varepsilon)$ and $\Omega_k(\tau, \varepsilon)$, we clarify the additional conditions which must be satisfied by operator $\Omega_k(\tau, \varepsilon)$ $\Omega_{k0} + \varepsilon\Omega_k(\tau, \varepsilon)$ which, in turn, is subject to conditions (S_1) and (S_2), in order /200

that the relationship (31.6) exist. It is easy to verify that for this purpose it is sufficient for the following equality to be satisfied:

$$\Omega_k(\tau, \varepsilon) = P_k \Omega_k(\tau, \varepsilon) P_k. \tag{31.9}$$

Indeed, in the case (31.9) equation (31.7) is satisfied by the operators $X_k(t, t_0, \varepsilon)$ and $P_k X_k(t, t_0, \varepsilon)$ in addition to $X_k(t, t_0, \varepsilon)P_k$:

$$P_k \frac{d}{dt} X_k(t, t_0, \varepsilon) = P_k \Omega_k P_k (P_k X_k) = \Omega_k(\tau, \varepsilon) P_k X_k(t, t_0, \varepsilon),$$

$$\frac{d}{dt} X_k(t, t_0, \varepsilon) P_k = \Omega_k(t, \varepsilon) X_k(t, t_0, \varepsilon) P_k.$$

If we assume that all of these solutions coincide when $t=t_0$:

$$X(t_0, t_0, \varepsilon) = P_k X_k(t_0, t_0, \varepsilon) = X_k(t_0, t_0, \varepsilon) P_k,$$

relationships (31.6) will be satisfied for any $\tau \in [0, \frac{L}{\varepsilon}]$ because according to Theorem 5.1, equation (31.7) has a unique solution which satisfies the assigned initial condition.

Thus we have become convinced that equality (31.9) provides the fulfillment of condition (31.6) for operators $X_k(t, t_0, \varepsilon)$; in other words, equation (31.7) is an equation in subspaces $\mathfrak{X}_k = P_k[\mathfrak{X}]$. Thus, by using an analogy with the finite case we can say that the order of these equations appears to be less than that of equation (31.1).

Now let us construct the operator $V(t, t_0, \varepsilon)$ which satisfies equation (31.1) at least formally. For this purpose we substitute expression (31.4) into (31.1) where $V(t, t_0, \varepsilon)$ has the form of (31.5).

Using representations (31.7) and (31.8), we obtain the equality

$$\sum_{k=0}^{n} \left[\sum_{s=1}^{\infty} \varepsilon^s \frac{dV_{ks-1}}{d\tau} + \sum_{s=0}^{\infty} \varepsilon^s \sum_{j=0}^{s} V_{kj}(\tau) \Omega_{ks-j}(\tau) \right] X_k = \tag{31.10}$$

$$= \sum_{k=0}^{n} \left[\sum_{s=0}^{\infty} \varepsilon^s A V_{ks}(\tau) + \sum_{s=1}^{\infty} \varepsilon^s \sum_{j=0}^{s-1} B_{s-j}(\tau) V_{kj}(\tau) \right] X_k.$$

If in equality (31.10) we separate the coefficients in front of various expressions of the type $\varepsilon^s X_k$, k=0, 1, 2, ..., n; s=0, 1, 2, ..., we find (taking into account requirement (31.6)) a series of recurrence relationships for determining the terms of expansion (31.8):

$$\frac{dV_{ks-1}}{d\tau}P_k + \sum_{j=0}^{s} V_{kj}(\tau)\Omega_{ks-j}(\tau)P_k = AV_{ks}(\tau)P_k + \sum_{j=0}^{s-1} B_{s-j}(\tau)V_{kj}(\tau)P_k \quad (31.11)$$

$(k = 0, 1, \ldots, n;\ s = 0, 1, 2, \ldots;\ V_{k-1}(\tau) \equiv 0).$

rom this we can determine all the quantities of interest to us. When s=0 we ave /201

$$V_{k0}(\tau)\Omega_{k0}(\tau)P_k = AV_{k0}(\tau)P_k. \quad (31.12)$$

iis relationship is satisfied if we let

$$V_{k0}(\tau) = P_k;\quad \Omega_{k0} = AP_k. \quad (31.13)$$

Obviously with this selection of operator $\Omega_{k0}(\tau)$, relationship (31.9) is atisfied on the elements from domain D(A).

When s=1 we obtain the following from system (31.11):

$$AV_{k1}(\tau)P_k - V_{k1}(\tau)AP_k = P_k\Omega_{k1}(\tau)P_k - B_1(\tau)P_k$$

· on the elements from D(A):

$$AV_{k1}(\tau)P_k - V_{k1}(\tau)P_kA = P_k\Omega_{k1}(\tau)P_k - B_1(\tau)P_k. \quad (31.14)$$

To solve equation (31.14) we utilize the results of Section 30, from which · follows that equation (31.14) can be solved under the condition

$$P_k[P_k\Omega_{k1}(\tau)P_k - B_1(\tau)P_k] = 0, \quad (31.15)$$

ich together with (31.9) makes it possible for us to determine the operator

$$\Omega_{k1}(\tau) = P_k\Omega_{k1}(\tau)P_k = P_kB_1(\tau)P_k. \quad (31.16)$$

In this case the unknown solution $V_{k1}(\tau)P_k$ has the form:

$$V_{k1}(\tau)P_k = \frac{1}{2\pi i}\int_{\gamma_k} R(\mu;\ H)H[\Omega_{k1}(\tau) - B_1(\tau)]P_kHR(\mu;\ H)\,d\mu, \quad (31.17)$$

ere

$$H = [A - \alpha I]^{-1},\quad \alpha \overline{\in} \sigma(A).$$

We now let

$$V_{k1}(\tau) = V_{k1}(\tau)P, \quad (31.18)$$

and note that according to Theorem 5.4

$$P_k V_{k1}(\tau) = 0. \tag{31.19}$$

If we consider relationship (31.11) for an arbitrary s>1 we obtain the equation

$$AV_{ks}(\tau) P_k - V_{ks}(\tau) P_k A = P_k \Omega_{ks}(\tau) P_k - T_{ks}(\tau) P_k, \tag{31.20}$$

where

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$$T_{ks}(\tau) = \sum_{j=0}^{s-1} B_{s-j}(\tau) V_{kj}(\tau) - \sum_{j=1}^{s-1} V_{kj}(\tau) \Omega_{ks-j}(\tau) - \frac{dV_{ks-1}}{d\tau}.$$

If we assume that all the quantities with numbers less than s are already known, the operator T_{ks} is known and equation (31.20) is solved in the same manner as for the case s=1.

From the conditions of solvability

$$P_k [P_k \Omega_{ks}(\tau) P_k - T_{ks}(\tau) P_k] = 0$$

we find

$$\begin{aligned} &\Omega_{ks}(\tau) = P_k \Omega_{ks}(\tau) P_k = P_k T_{ks}(\tau) P_k = \\ &= \sum_{j=0}^{s-1} P_k B_{s-j}(\tau) V_{kj}(\tau) - P_k \frac{dV_{ks-1}}{d\tau} \\ &(k = 0, 1, \ldots, n;\ s = 1, 2, \ldots). \end{aligned} \tag{31.21}$$

then

$$\begin{aligned} V_{ks}(\tau) = V_{ks}(\tau) P_k &= \frac{1}{2\pi i} \int_{\gamma_k} R(\mu;\ H) H [\Omega_{ks}(\tau) - \\ &- T_{ks}(\tau)] P_k H R(\mu;\ H) d\mu \\ &(k = 0, 1, \ldots, n;\ s = 1, 2, \ldots). \end{aligned} \tag{31.22}$$

The above approach may be used to determine the operators $\Omega_{ks}(\tau)$ and $V_{ks}(\tau)$ with sufficiently large numbers s. We should note that the expression for $V_{ks}(\tau)$ contains a derivative $\frac{dV_{ks-1}}{d\tau}$.

By using (31.21) and (31.22), it is easy to show that in the case when the operators $B_{p-j}(\tau)$ $(p \geq 1, 2, \ldots, p-1)$ can be differentiated j+2 times with respect

to τ, it is possible to compute the unknown solutions $\Omega_{ks}(\tau)$ and $V_{ks}(\tau)$ for $0 \leq s \leq p$. These solutions will be continuously differentiable $(p-s+2)$ times with respect to τ.

Thus we have described an algorithm for constructing the terms of expansion (31.8) for operators $V_k(\tau, \varepsilon)$ and $\Omega_k(\tau, \varepsilon)$. If it were possible to determine an infinite number of terms in these expansions and then to find the resolving operator $X_k(t, t_0, \varepsilon)$ of equations (31.7), the constructed function

$$x(t, \varepsilon) = V(t, t_0, \varepsilon) x_0,$$

where $V(t, t_0, \varepsilon)$ has the structure of (31.5), would represent the formal solution of problem (31.1) and (31.2).

However, actually it always becomes necessary to limit ourselves to a finite number of steps in the algorithm described above. Therefore we shall consider the operator

$$V_{k_*}^{(p)}(t, t_0, \varepsilon) = \sum_{k=0}^{n} V_k^{(p)}(\tau, \varepsilon) X_k^{(p)}(t, t_0, \varepsilon), \qquad (31.23)$$ /203

where

$$V_k^{(p)}(\tau, \varepsilon) = \sum_{s=0}^{p} \varepsilon^s V_{ks}(\tau), \qquad (31.24)$$

while $X_k^{(p)}(t, t_0, \varepsilon)$ are operators which satisfy (in the sense defined earlier) the differential equations

$$\frac{dX_k^{(p)}(t, t_0, \varepsilon)}{dt} = \Omega_k^{(p)}(\tau, \varepsilon) X_k^{(p)}(t, t_0, \varepsilon) \qquad (k = 0, 1, 2, \ldots, n) \qquad (31.25)$$

with operational coefficients

$$\Omega_k^{(p)}(\tau, \varepsilon) = \sum_{s=0}^{p} \varepsilon^s \Omega_{ks}(\tau) = \Omega_{k0} + \varepsilon \overline{\Omega}_k^{(p)}(\tau, \varepsilon),$$
$$\overline{\Omega}_k^{(p)}(\tau, \varepsilon) = \sum_{s=1}^{p} \varepsilon^{s-1} \Omega_{ks}(\tau). \qquad (31.26)$$

According to our previous considerations the operator $V^{(p)}(t, t_0, \varepsilon)$ has a continuous derivative with respect to t. Consequently we can differentiate it

by means of relationship (31.11). As a result we obtain the following equation for $V^{(p)}(t, t_0, \varepsilon)$:

$$\frac{dV^{(p)}(t, t_0, \varepsilon)}{dt} = [A + \varepsilon B^{(p)}(\tau, \varepsilon)] V^{(p)}(t, t_0, \varepsilon) + \varepsilon^{p+1} \Phi_p(t, t_0, \varepsilon), \tag{31.27}$$

where

$$B^{(p)} = \sum_{s=1}^{p} \varepsilon^{s-1} B_s(\tau),$$

$$\Phi_p(t, t_0, \varepsilon) = \sum_{k=0}^{n} \left[\frac{dV_{kp}(\tau)}{d\tau} + \sum_{s=p+1}^{2p} \varepsilon^{s-p-1} \sum_{j=s-p}^{p} (V_{ks-j}(\tau) \Omega_{kj}(\tau) - \right. \tag{31.28}$$

$$\left. - B_j(\tau) V_{ks-j}(\tau) \right] X_k^{(p)}(t, t_0, \varepsilon) \equiv \sum_{k=0}^{n} \Phi_k^{(p)}(\tau, \varepsilon) X_k^{(p)}(t, t_0, \varepsilon).$$

A more detailed investigation of operator $V^{(p)}(t, t_0, \varepsilon)$ will be carried out in the next section.

Section 32. Proof of Asymptotic Convergence

To clarify the properties of operator $V^{(p)}(t, t_0, \varepsilon)$, we consider the operators $X_k^{(p)}(t, t_0, \varepsilon)$ $(k=0, 1, \ldots, n)$ which establish the solution of equations /20

$$\frac{dX_k^{(p)}(t, t_0, \varepsilon)}{dt} = [\Omega_{k0}(\tau) + \varepsilon \bar{\Omega}_k^{(p)}(\tau, \varepsilon)] X_k^{(p)}(t, t_0, \varepsilon) \tag{32.1}$$

$$(k = 0, 1, 2, \ldots, n),$$

assigned in subspaces $P_k[\mathcal{X}]$.

As a result of our construction the coefficients $\Omega_k^{(p)}(\tau, \varepsilon) = \Omega_{k0} + \varepsilon \bar{\Omega}_k^{(p)}(\tau, \varepsilon)$ satisfy the conditions of Theorem 5.1. Therefore, we can maintain that the resolving operator $X_k^{(p)}(t, t_0, \varepsilon)$ of equations (32.1) is bounded

$$\| X_k^{(p)}(t, t_0, \varepsilon) \| \leqslant e^{\varepsilon N (t - t_0)} \leqslant e^{NL},$$

$$[N = \max_k \{N_k\}, \qquad N_k = \max_{0 \leqslant \tau \leqslant L, 0 < \varepsilon \leqslant \varepsilon_0} \| \bar{\Omega}_k^{(p)}(\tau, \varepsilon) \|, \tag{32.2}$$

$$k = 0, 1, 2, \ldots, n],$$

and that for the function $X_k^{(p)}(t,s,\varepsilon)y(y\in D(A))$ of parameter s $(0\le s\le t\le \frac{L}{\varepsilon})$ the following relationship exists

$$\frac{dX_k^{(p)}(t,s,\varepsilon)}{ds}y = -X_k^{(p)}(t,s,\varepsilon)[\Omega_{k0}+\varepsilon\bar{\Omega}_k^{(p)}(\sigma,\varepsilon)]y, \qquad \sigma=\varepsilon s. \tag{32.3}$$

Thus, in view of the properties of operators $X_k^{(p)}(t,t_0,\varepsilon)$ and the properties of operators $V_{kj}(\tau),\Omega_{kj}(\tau)(k=0,\ 1,\ \ldots,\ n;j=1,\ 2,\ \ldots,\ p)$, which we have constructed in the preceding section, it follows that the operators

$$\begin{aligned} V^{(p)}(t,t_0,\varepsilon) &= \sum_{k=0}^{n} V_k^{(p)}(\tau,\varepsilon)X_k^{(p)}(t,t_0,\varepsilon),\\ \Phi_p(t,t_0,\varepsilon) &= \sum_{k=0}^{n} \Phi_k^{(p)}(\tau,\varepsilon)X_k^{(p)}(t,t_0,\varepsilon) \end{aligned} \tag{32.4}$$

(see equation (31.27)) are bounded on the segment $0\le t_0\le t\le \frac{L}{\varepsilon}$, while the operational functions $V^{(p)}(t,t_0,\varepsilon)y,\Phi^{(p)}(t,t_0,\varepsilon)y,y\in D(A)$ are continuously differentiable with respect to t and t_0.

It is easy to see that the following equation exists for the function $V^{(p)}(t,t_0,\varepsilon)y,y\in D(A)$

$$\frac{dV^{(p)}(t,t_0,\varepsilon)}{dt}y = [A+\varepsilon B(\tau,\varepsilon)]V^{(p)}(t,t_0,\varepsilon)y + \varepsilon^{p+1}\Psi(t,t_0,\varepsilon)y, \tag{32.5}$$

where

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$$\Psi(t,t_0,\varepsilon) = \Phi_p(t,t_0,\varepsilon) - \bar{B}^{(p)}(\tau,\varepsilon)V^{(p)}(t,t_0,\varepsilon),$$

$$\bar{B}^{(p)}(\tau,\varepsilon) = \sum_{s=1}^{\infty}\varepsilon^{s-1}B_{s+p}(\tau)$$

(compare (32.5) and (31.27)).

In other words, $x^{(p)}(t,\varepsilon)=V^{(p)}(t,t_0,\varepsilon)y,y\in D(A)$ satisfies the initial equation

$$\frac{dx}{dt} = [A+\varepsilon B(\tau,\varepsilon)]x(t,\varepsilon) \tag{32.6}$$

with an accuracy determined by an expression of order $O(\varepsilon^{p+1})$ because, according to the established properties of operators $\Phi_p(t,t_0,\varepsilon)V^{(p)}(t,t_0,\varepsilon)$ and conditions (S_2) imposed on $B(\tau,\varepsilon)$, the operator $\Psi(t,t_0,\varepsilon)$ is bounded on the segment $0\le t_0\le t\le \frac{L}{\varepsilon}$ and is subject to the following estimate

$$\| \Psi (t, t_0, \varepsilon) \| \leqslant C_1,$$

where C_1 is a constant which is independent of ε.

It is now easy to show that the operator $V^{(p)}(t, t_0, \varepsilon)$ which we have constructed converges asymptotically to the resolving operator $V(t, t_0, \varepsilon)$ of equation (32.6). As shown earlier, $V(t, t_0, \varepsilon)$ has the following properties:

$$\frac{dV(t, t_0, \varepsilon)}{dt} y = [A + \varepsilon B(\tau, \varepsilon)] V(t, t_0, \varepsilon) y, \qquad y \in D(A), \tag{32.7}$$
$$V(t_0, t_0, \varepsilon) = I, \qquad \| V(t, t_0, \varepsilon) \| \leqslant e^{M_1 L}.$$

We introduce the difference

$$Z(t, t_0 \varepsilon) = V^{(p)}(t, t_0, \varepsilon) - V(t, t_0, \varepsilon). \tag{32.8}$$

According to (32.5) and (32.7), the operator $Z(t, t_0, \varepsilon)$ satisfies the equation

$$\frac{dZ(t, t_0, \varepsilon)}{dt} y = [A + \varepsilon B(\tau, \varepsilon)] Z(t, t_0, \varepsilon) y + \varepsilon^{p+1} \Psi (t, t_0, \varepsilon) y.$$

and

$$Z(t_0, t_0, \varepsilon) = V^{(p)}(t_0, t_0, \varepsilon) - I.$$

By using the results of Theorem 5.2, we can maintain that the following representation exists for operator $Z(t, t_0, \varepsilon)$ on the elements $y \in D(A)$:

$$Z(t, t_0, \varepsilon) = V(t, t_0, \varepsilon) Z(t_0, t_0, \varepsilon) + \varepsilon^{p+1} \int_{t_0}^{t} V(t, s, \varepsilon) \Psi (s, t_0, \varepsilon) ds. \tag{32.9}$$

If at the initial instant of time ($t = t_0$) the inequality /206

$$\| V^{(p)}(t_0, t_0, \varepsilon) - I \| \leqslant C_0 \varepsilon^p, \tag{32.10}$$

is satisfied then it is easy to obtain the following estimate from equality (32.9) if we take into account (32.8):

$$\| V^{(p)}(t, t_0, \varepsilon) - V(t, t_0, \varepsilon) \| \leqslant C \varepsilon^p. \tag{32.11}$$

(Here we have used the restricted nature of operators $V(t, t_0, \varepsilon)$ and $\Psi(t, t_0, \varepsilon)$, which was established earlier.)

Thus we have proved the following theorem.

Theorem 5.5. If the operators A and $B(\tau,\varepsilon)$ satisfy conditions (S_1) and (S_2), from the estimate

$$\| V^{(p)}(t_0, t_0, \varepsilon) - I \| \leqslant C_0 \varepsilon^p$$

for operators $V^{(p)}(t, t_0, \varepsilon)$, the following estimate follows:

$$\| V^{(p)}(t, t_0, \varepsilon) - V(t, t_0, \varepsilon) \| \leqslant C\varepsilon^p \qquad \left(0 \leqslant t_0 \leqslant t \leqslant \frac{L}{\varepsilon}\right),$$

where C is some constant which is independent of ε.

Section 33. The Asymptotic Solution of the Nonhomogeneous Equation

Let us now consider the nonhomogeneous equation

$$\frac{dx}{dt} = [A + \varepsilon B(\tau, \varepsilon)]\, x + f(\tau, \varepsilon) e^{\theta(t,\varepsilon)}, \tag{33.1}$$

where the operators A and $B(\tau,\varepsilon)$ are the same as those in Section 31 while the vector function $f(\tau,\varepsilon)$ is continuously differentiable with respect to $\tau \in [0,L]$ and can be represented in the following form:

$$f(\tau, \varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s f_s(\tau), \qquad \| f(\tau, \varepsilon) \| \leqslant h. \tag{33.2}$$

In addition let us assume that the complex function $\theta(t,\varepsilon)$ satisfies the conditions

$$\operatorname{Re} \theta(t, \varepsilon) \leqslant 0, \qquad \frac{d\theta}{dt} = \nu(\tau), \tag{33.3}$$

where $\nu(\tau)$ is a function which is continuous in $[0,L]$.

As follows from Theorem 5.2, the solution of equation (33.1), which satisfies condition $x(0)=x_0$, has the form

$$x(t, \varepsilon) = V(t, 0, \varepsilon) x_0 + \int_0^t V(t, s, \varepsilon) f(\sigma, \varepsilon) e^{\theta(s,\varepsilon)} ds, \qquad \sigma = \varepsilon s. \tag{33.4}$$

To construct an approximation to this solution we introduce the operator /207

$$V^{(p)}(t, t_0, \varepsilon) = \sum_{j=0}^{n} V_j^{(p)}(\tau, \varepsilon) X_j^{(p)}(t, t_0, \varepsilon), \tag{33.5}$$

where $V_j^{(p)}(\tau, \varepsilon)$ and $X_j^{(p)}(t, t_0, \varepsilon)$ are operators constructed in Section 31 and in this case the initial conditions for $X_j^{(p)}(t, t_0, \varepsilon)$ are selected in such a way that

$$\| V^{(p)}(t_0, t_0, \varepsilon) - I \| \leqslant C_0 \varepsilon^p. \tag{33.6}$$

We shall now let

$$x^{(p)}(t, \varepsilon) = V^{(p)}(t, 0, \varepsilon) x_0 + \int_0^t V^{(p)}(t, s, \varepsilon) f^{(p)}(\sigma, \varepsilon) e^{\theta(s,\varepsilon)} ds, \tag{33.7}$$

where

$$f^{(p)}(\tau, \varepsilon) = \sum_{s=0}^{p} \varepsilon^s f_s(\tau),$$

and evaluate the difference

$$y(t, \varepsilon) = x(t, \varepsilon) - x^{(p)}(t, \varepsilon).$$

According to (33.4) and (33.5) we have

$$\begin{aligned} x(t, \varepsilon) - x^{(p)}(t, \varepsilon) = [V(t, 0, \varepsilon) - V^{(p)}(t, 0, \varepsilon)] x_0 + \\ + \int_0^t [V(t, s, \varepsilon) - V^{(p)}(t, s, \varepsilon)] f^{(p)}(\sigma, \varepsilon) e^{\theta(s,\varepsilon)} ds + \\ + \varepsilon^{p+1} \int_0^t V(t, s, \varepsilon) \bar{f}^{(p)} e^{\theta(s,\varepsilon)} ds, \end{aligned} \tag{33.8}$$

where

$$\bar{f}^{(p)}(\tau, \varepsilon) = \sum_{s=1}^{\infty} \varepsilon^{s-1} f_{s+p}(\tau).$$

Using the results of the preceding sections, it is easy to estimate the element $y(t, \varepsilon)$. Indeed,

$$\begin{aligned} \| y(t, \varepsilon) \| = \| x(t, \varepsilon) - x^{(p)}(t, \varepsilon) \| < C \| x_0 \| \varepsilon^p + \\ + Ch \varepsilon^p \frac{L}{\varepsilon} + \varepsilon^{p+1} e^{M_1 L} h \frac{L}{\varepsilon} \leqslant C_2 \varepsilon^{p-1}, \end{aligned}$$

where

$$C_2 = ChL + C \| x_0 \| \varepsilon_0 + \varepsilon_0 h L e^{M_1 L}.$$

Thus, in the case of the nonhomogeneous equation (33.1) the approximate solution (33.7) also has an asymptotic nature, i.e., the following estimate /208 is valid for solution $x^{(p)}(t, \varepsilon)$:

$$\| x(t, \varepsilon) - x^{(p)}(t, \varepsilon) \| \leqslant C_2 \varepsilon^{p-1}, \tag{33.9}$$

if at the initial instant of time (t=0) the following inequality exists:

$$\| x(0, \varepsilon) - x^{(p)}(0, \varepsilon) \| \leqslant C_0 \varepsilon^p.$$

If we write expression (33.7) in a more detailed form

$$x^{(p)}(t, \varepsilon) = \sum_{k=0}^{n} V_k^{(p)}(\tau, \varepsilon) \left[X_k^{(p)}(t, 0, \varepsilon) x_0 + \int_0^t X_k^{(p)}(t, s, \varepsilon) f^{(p)}(\sigma, \varepsilon) e^{\theta(s, \varepsilon)} ds \right], \tag{33.10}$$

it then becomes clear that when conditions $P_k f(\sigma, \varepsilon)=0, P_k x_0=0$ are satisfied for certain subscripts k=0, 1, ..., n the terms with the corresponding numbers k will not be contained in expression (33.10).

Let us now show that the estimate (33.9) for the approximate solution to the nonhomogeneous equation (33.1) may be improved in the case of "nonresonance." As before, this case is characterized by the fact that the values of function $v(\tau)$ for $\tau \in [0, L]$ cannot coincide with the spectrum points of operator A.

Lemma 5.2. Let us assume that the conditions relative to operators $A, B(\tau, \varepsilon)$, the vector function $f(\tau, \varepsilon)$, and the scalar function $\theta(t, \varepsilon)$ formulated earlier are satisfied. In addition let us assume that operators $X_k^{(p)}(t, t_0, \varepsilon)$ are selected in such a way that

$$\| V^{(p)}(t, t, \varepsilon) - I \| \leqslant C_3 \varepsilon^{p+1}. \tag{33.11}$$

Then, in the case of "nonresonance" the approximate solution $x^{(p)}(t, \varepsilon)$ satisfies the equation

$$\frac{dx^{(p)}}{dt} = [A + \varepsilon B(\tau, \varepsilon)] x^{(p)} + f(\tau, \varepsilon) e^{\theta(t, \varepsilon)} + \varepsilon^{p+1} \varphi(t, \varepsilon),$$

where the vector function $\varphi(t, \varepsilon)$ is uniformly bounded with respect to t and ε in the domain $0 \leq t \leq \frac{L}{\varepsilon}, 0 < \varepsilon \leq \varepsilon_0$.

Proof. We differentiate with respect to t the approximate solution $x^{(p)}(t, \varepsilon)$ determined by equation (33.7). Then, due to the method used to construct expression (33.7) we obtain the following equation for $x^{(p)}(t, \varepsilon)$:

$$\frac{dx^{(p)}}{dt} = [A + \varepsilon B(\tau, \varepsilon)]\, x^{(p)} + f(\tau, \varepsilon)\, e^{\theta(t,\varepsilon)} +$$
$$+ g(t, \varepsilon) + \varepsilon^{p+1} \int_0^t \Psi(t, s, \varepsilon)\, f^{(p)}(\sigma, \varepsilon)\, e^{\theta(s,\varepsilon)}\, ds, \qquad (33.12)$$

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where

$$g(t, \varepsilon) = \varepsilon^{p+1} [\Psi(t, t_0, \varepsilon)\, x_0 - \overline{f}^{(p)}(\tau, \varepsilon)\, e^{\theta(t,\varepsilon)}] +$$
$$+ [V^{(p)}(t, t, \varepsilon) - I]\, f^{(p)}(\tau, \varepsilon)\, e^{\theta(t,\varepsilon)}.$$

In view of the boundedness of operator $\Psi(t, t_0, \varepsilon)$, shown in Section 32 and the propositions of the given lemma, we can maintain that the vector function $g(t, \varepsilon)$ is of the order $O(\varepsilon^{p+1})$, i.e.,

$$\| g(t, \varepsilon) \| \leqslant C_4 \varepsilon^{p+1}, \qquad C_4 = \text{const.} \qquad (33.13)$$

It is easy to verify that in the considered case of "nonresonance" the last term in the right side of equation (33.12) is of the order $O(\varepsilon^{p+1})$, i.e., for the integral

$$I = \int_0^t \Psi(t, s, \varepsilon)\, f^{(p)}(\sigma, \varepsilon)\, e^{\theta(s,\varepsilon)} ds \qquad (33.14)$$

the following estimate is valid:

$$\| I \| < C_5 = \text{const.} \qquad (33.15)$$

To show this we use the representation

$$\Psi(t, s, \varepsilon) = \Phi_p(t, s, \varepsilon) - \overline{B}^{(p)}(\tau, \varepsilon)\, V^{(p)}(t, s, \varepsilon)$$

and equations (32.4). Then expression (33.14) may be rewritten in the form

$$I = \sum_{k=0}^{n} Q_k(\tau, \varepsilon) \int_0^t X_k^{(p)}(t, s, \varepsilon)\, f^{(p)}(\sigma, \varepsilon)\, e^{\theta(s,\varepsilon)} ds, \qquad (33.16)$$

where

$$Q_k(\tau, \varepsilon) = \Phi_k^{(p)}(\tau, \varepsilon) - \overline{B}^{(p)}(\tau, \varepsilon)\, V_k^{(p)}(\tau, \varepsilon).$$

We note that in accordance with the preceding considerations, operators $Q_k(\tau, \varepsilon)$ are uniformly bounded.

Let us now investigate the integral in expression (33.16). For this purpose we introduce the following designations:

$$\int_0^t X_k^{(p)}(t, s, \varepsilon) f^{(p)}(\sigma, \varepsilon) e^{\theta(s,\varepsilon)} ds = e^{\theta(t,\varepsilon)} \int_0^t Y_k(t, s, \varepsilon) f^{(p)}(\sigma, \varepsilon)\, ds, \tag{33.17}$$

where

$$Y_k(t, s, \varepsilon) = e^{-[\theta(t,\varepsilon)-\theta(s,\varepsilon)]} X_k^{(p)}(t, s, \varepsilon). \tag{33.18}$$

According to the properties of operators $X_k^{(p)}(t, s, \varepsilon)$ established earlier (see (32.3)), the following equation is valid: /210

$$\frac{dY_k(t, s, \varepsilon)}{ds} = -Y_k(t, s, \varepsilon)[\Omega_{k0} - \nu(\sigma) I] - \varepsilon Y_k(t, s, \varepsilon)\overline{\Omega}_k^{(p)}(\sigma, \varepsilon). \tag{33.19}$$

Since according to the proposition the function $\nu(\sigma), \sigma=\varepsilon s$ does not coincide with the spectrum points of operator A while $\Omega_{k0}=AP_k$ (according to the construction), a bounded reciprocal operator $[\Omega_{k0}-\nu(\sigma)I]^{-1}$ exists for any k=0, 1, 2, ..., n.

As a result, from equation (33.19) we have

$$\begin{aligned} Y_k(t, s, \varepsilon) = &-\frac{dY_k}{ds}[\Omega_{k0} - \nu(\sigma) I]^{-1} - \\ &- \varepsilon Y_k(t, s, \varepsilon)\overline{\Omega}_k^{(p)}(\sigma, \varepsilon)[\Omega_{k0} - \nu I]^{-1}. \end{aligned} \tag{33.20}$$

Substituting expression $Y_k(t, s, \varepsilon)$ into relationship (33.17) and integrating in parts, we find

$$\begin{aligned} &\int_0^t X_k^{(p)}(t, s, \varepsilon) f^{(p)}(\sigma, \varepsilon) e^{\theta(s,\varepsilon)} ds = \\ &= -e^{\theta(t,\varepsilon)} X_k^{(p)}(t, t, \varepsilon)[\Omega_{k0} - \nu(\tau) I]^{-1} f^{(p)}(\tau, \varepsilon) + \\ &+ e^{\theta(0,\varepsilon)} X^{(p)}(t, 0, \varepsilon)[\Omega_{k0} - \nu(0) I]^{-1} f^{(p)}(0, \varepsilon) + \varepsilon \int_0^t e^{\theta(s,\varepsilon)} X_k^{(p)}(t, s, \varepsilon) \times \\ &\times \left\{ \frac{d}{d\sigma}[(\Omega_{k0} - \nu(\sigma) I)^{-1} f^{(p)}(\sigma, \varepsilon)] - \right. \\ &\left. - \overline{\Omega}^{(p)}(\sigma, \varepsilon)[\Omega_{k0} - \nu(\sigma) I]^{-1} f^{(p)}(\sigma, \varepsilon) \right\} ds. \end{aligned}$$

From the last relationship it is easy to obtain the following estimate

$$\left\| \int_0^t X_k^{(p)}(t, s, \varepsilon) f^{(p)}(\sigma, \varepsilon) e^{\theta(s,\varepsilon)} ds \right\| \leqslant \beta_k = \text{const}. \tag{33.21}$$

Since inequality (33.21) in this case exists for all k(k=0, 1, 2, ..., n) the validity of inequality (33.15) is now obvious.

Combining the results which we have obtained, we can represent equation (33.12) in the form

$$\frac{dx^{(p)}}{dt} = [A + \varepsilon B(\tau, \varepsilon)]\, x^{(p)} + f(\tau, \varepsilon)\, e^{\theta(t,\varepsilon)} + \varepsilon^{p+1} \varphi(t, \varepsilon), \tag{33.22}$$

where $\varphi(t, \varepsilon)$ is a uniformly bounded function: /211

$$\| \varphi(t, \varepsilon) \| \leqslant C_6 = \text{const.}$$

In other words, the approximate solution $x^{(p)}(t, \varepsilon)$, when the conditions of Lemma 5.2 are satisfied, satisfies the given equation (33.1) with an accuracy determined by a quantity of order $O(\varepsilon^{p+1})$. Lemma 5.2 has been proven.

We let

$$z(t, \varepsilon) = x^{(p)}(t, \varepsilon) - x(t, \varepsilon) \tag{33.23}$$

be the difference between the approximate and accurate solution of equation (31.1). Then, on the basis of (33.1) and (33.22) we obtain the following equation for $z(t, \varepsilon)$:

$$\frac{dz}{dt} = [A + \varepsilon B(\tau, \varepsilon)]\, z + \varepsilon^{p+1} \varphi(t, \varepsilon). \tag{33.24}$$

In this case, according to (33.7) and (33.4) the initial condition $z_0 = z(0, \varepsilon)$ has the form

$$z_0 = x^{(p)}(0, \varepsilon) - x(0, \varepsilon) = [V^{(p)}(0, 0, \varepsilon) - I]\, x_0. \tag{33.25}$$

According to Theorem 5.2, we can write

$$z(t, \varepsilon) = V(t, 0, \varepsilon)\, z_0 + \varepsilon^{p+1} \int_0^t V(t, s, \varepsilon)\, \varphi(s, \varepsilon)\, ds,$$

from which the following estimate follows immediately if the conditions of Lemma 5.2 are satisfied:

$$\| x^{(p)}(t, \varepsilon) - x(t, \varepsilon) \| = \| z(t, \varepsilon) \| \leqslant C_7 \varepsilon^p. \tag{33.26}$$

Thus we can maintain that when we use equation (33.7) to construct the approximate solution $x^{(p)}(t, \varepsilon)$ for the nonhomogeneous equation (33.1) the following theorem is valid.

Theorem 5.6. Let the operators A and $B(\tau, \varepsilon)$ satisfy conditions (S_1) and

(S_2) and let the vector function $f(\tau,\varepsilon)$ be continuously differentiable with

respect to $\tau\in[0,L]$ and let us assume that representation (33.2) is permissible. In addition we assume that the function $\theta(t,\varepsilon)$ satisfies conditions (33.3).

Then in the construction of the approximation $x^{(p)}(t,\varepsilon)$ to the exact solution of equation (33.1), when conditions (33.6) are satisfied, we have the following estimate

$$\| x^{(p)}(t,\varepsilon) - x(t,\varepsilon) \| \leqslant C_2\varepsilon^{p-1},$$

while in the case of "nonresonance" and condition (33.11) there is a better estimate

$$\| x^{(p)}(t,\varepsilon) - x(t,\varepsilon) \| \leqslant C_7\varepsilon^{p}.$$

Section 34. The Direct Construction of the Particular Solution for the Nonhomogeneous Equation

As shown in Section 33, (see equations (33.4), (33.7)), a particular solution of equation /212

$$\frac{dx}{dt} = [A + \varepsilon B(\tau,\varepsilon)]x + f(\tau,\varepsilon)e^{\theta(t,\varepsilon)} \qquad (34.1)$$

is expressed rather simply in terms of the general solution for the corresponding homogeneous equation. However, sometimes the construction of the general solution meets substantial difficulties or becomes generally impossible. In addition to this we may require a particular solution which has the same nature as the free term of equation (34.1). In this case the method described below for the direct construction of a particular asymptotic solution for equation (34.1) becomes useful.

In utilizing this method we should distinguish between two known cases:

1) the case of "resonance" in which the function $\nu(\tau)=\frac{d\theta}{dt}$ for certain

values $\tau\in[0,L]$ coincides with the points of one or several but not all of the spectral sets for operator A;

2) the case of "nonresonance" in which the function $\nu(\tau)$ does not have values on the segment $[0,L]$ which are common to the spectrum of operator A.

Let us indicate the method of constructing the formal particular solution of equation (34.1) in the first case. We shall assume that $\nu(\tau)$, for certain τ from $[0,L]$, coincides with the points of the spectral set σ_k. (We recall that the spectrum $\sigma(A)$ of operator A has the following structure: $\sigma(A)=\bigcup_0^n \sigma_j$; consequently, $0\leq k\leq n$.)

Under the assumptions which have been made (taking into account conditions formulated in Section 33) we reduce the particular solution of equation (34.1) to the form

$$x(t, \varepsilon) = [V_k(\tau, \varepsilon)\xi_k(t, \varepsilon) + h(\tau, \varepsilon)]\, e^{\theta(t,\varepsilon)}, \tag{34.2}$$

where $V_k(\tau, \varepsilon)$ is some operator; $h(\tau, \varepsilon)$ is some vector from the space $\mathfrak{X}$; and $\xi_k(t, \varepsilon)$ is the solution of equation

$$\frac{d\xi_k}{dt} = [\Omega_k(\tau, \varepsilon) - \nu(\tau) I]\,\xi_k + b_k(\tau, \varepsilon), \tag{34.3}$$

determined in the subspace $P_k[\mathfrak{X}]$, i.e.,

$$P_k\xi_k(t, \varepsilon) = \xi_k(t, \varepsilon). \tag{34.4}$$

In this case, as usual, it is assumed that operators $V_k(\tau, \varepsilon), \Omega_k(\tau, \varepsilon)$ and vectors $h(\tau, \varepsilon), b_k(\tau, \varepsilon)$ permit expansions in powers of the parameter ε which have been encountered many times before. /213

In proceeding with the construction of such expansions we note that to satisfy conditions (34.4) it is sufficient to require that

$$P_k\Omega_k(\tau, \varepsilon)P_k = \Omega_k(\tau, \varepsilon), \qquad P_k b_k(\tau, \varepsilon) = b_k(\tau, \varepsilon) \tag{34.5}$$

(Equalities (34.5) follow from the corresponding considerations of Section 31.)

Now let us substitute (34.2) into equation (34.1) and take into account equalities (34.3) and (34.4). By equating separately, in both sides of the resulting identity, the expressions in front of $e^{\theta(t,\varepsilon)}\xi_k(t, \varepsilon)$ and the coefficients of $e^{\theta(t,\varepsilon)}$, we find two relationships which can be used to determine the unknown operators and vectors:

$$\varepsilon\frac{dV_k}{d\tau}P_k + V_k(\tau, \varepsilon)\Omega_k(\tau, \varepsilon)P_k = [A + \varepsilon B(\tau, \varepsilon)]\,V_k P_k, \tag{34.6}$$

$$[A - \nu(\tau) I]\,h(\tau, \varepsilon) = \varepsilon\frac{dh}{d\tau} + V_k(\tau, \varepsilon)b_k(\tau, \varepsilon) - f(\tau, \varepsilon) - \varepsilon B(\tau, \varepsilon)h(\tau, \varepsilon). \tag{34.7}$$

Since relationship (34.6) coincides completely with the corresponding equality (31.10), we shall not consider at this time the construction of terms in the expansions of operators $V_k(\tau, \varepsilon)$ and $\Omega_k(\tau, \varepsilon)$. These are constructed in exactly the same way as shown in Section 31.

We shall say a few words about finding the terms of expansions for vectors $h(\tau,\varepsilon)$ and $b_k(\tau,\varepsilon)$, assuming that in equality (34.7) the operator $V_k(\tau,\varepsilon)$ is already known. By separating the coefficients in front of the successive powers of parameter ε in expressions (34.7), we obtain the following recurrence relationships which can be used to determine the unknown $h_s(\tau)$ and $b_{ks}(\tau)$ $(s=0, 1, 2, \ldots)$:

$$[A-\nu(\tau)I]h_s(\tau)=b_{ks}(\tau)+T_s(\tau)\equiv F_s(\tau), \qquad s=0,1,2,\ldots, \tag{34.8}$$

where

$$T_s(\tau)=\frac{dh_{s-1}}{d\tau}+\sum_{m=0}^{s-1}V_{ks-m}(\tau)b_{km}(\tau)-f_s(\tau)-B_s(\tau). \tag{34.9}$$

When s=0 we obtain the following from relationship (34.8):

$$[A-\nu(\tau)I]h_0(\tau)=b_{k0}(\tau)-f_0(\tau)\equiv F_0(\tau). \tag{34.10}$$

In solving equation (34.10) we use the same approach as in Section 31; specifically, we find $b_{k0}(\tau)$ from the condition that equation (34.10) is solvable:

$$P_kF_0(\tau)=P_k[b_{k0}(\tau)-f_0(\tau)]=0,$$

i.e., according to (34.5), /214

$$b_{k0}(\tau)=P_kf_0(\tau). \tag{34.11}$$

Now we find the unknown vector $h_0(\tau)$ in accordance with the results of Section 30 by means of the equation

$$h_0(\tau)=\frac{1}{2\pi i}\int_{\gamma_k}\frac{R(\mu;H)HF_0(\tau)}{1-\mu[\nu(\tau)-\alpha]}d\mu, \tag{34.12}$$

where $H=(A-\alpha I)^{-1}$ is a bounded operator constructed in Section 30, and γ_k is an arc which surrounds the spectral set $\sigma_k'=\frac{1}{\sigma_k-\alpha}$ of operator H.

By using a different form for the solution of equation (34.10), specifically,

$$h_0(\tau)=-\frac{1}{2\pi i}\int_{\tilde{\gamma}_k}\frac{R(\mu;H)HF_0(\tau)}{1-\mu[\nu(\tau)-\alpha]}d\mu,$$

where $\tilde{\gamma}_k$ contains the spectrum $\sigma(H)-\sigma_k'$, we can represent the solution in the form

$$h_0(\tau)=\left[H-\frac{1}{\nu(\tau)-\alpha}\right]^{-1}[I-P_k]Hf_0.*$$

In the same way we solve equation (34.7) for the subsequent values $s\geq 1$.

In particular we find the vector $b_{ks}(\tau)$ from the condition

$$P_kF_s(\tau)\equiv P_k[b_{ks}(\tau)+T_s(\tau)]=0,$$

i.e.,

$$b_{ks}(\tau)=P_k\left[f_s(\tau)+B_s(\tau)-\frac{dh_{s-1}}{d\tau}\right] \qquad (s=1,2,\ldots), \tag{34.13}$$

while we determine the vector $h_s(\tau)$ by means of the equation

$$h_s(\tau)=\frac{1}{2\pi i}\int_{\gamma_k}\frac{R(\mu;H)HF_s(\tau)}{1-\mu[\nu(\tau)-\alpha]}d\mu \qquad (s=1,2,\ldots). \tag{34.14}$$

Thus we can find any term in the expansion for the unknown operators $V_k(\tau,\varepsilon),\Omega_k(\tau,\varepsilon)$ and vectors $h(\tau,\varepsilon),b_k(\tau,\varepsilon)$, and consequently we can construct a formal particular solution for the nonhomogeneous equation (34.1) in the case of "resonance."

When "resonance" is absent the particular solution of equation (34.1) is sought, as usual, in the form /215

$$x(t,\varepsilon)=h(\tau,\varepsilon)e^{\theta(t,\varepsilon)}.$$

For the homogeneous equation

$$\frac{dx}{dt}=[A+\varepsilon B(\tau,\varepsilon)]x$$

the particular solution corresponding to the spectral set σ_k is constructed by means of the equation

$$x(t,\varepsilon)=V_k(\tau,\varepsilon)\xi_k(t,\varepsilon),$$

---

* We note that the reciprocal operator $\left[H-\frac{1}{\nu(\tau)-\alpha}\right]^{-1}$, which does not exist in the entire space, can be determined, however, in the subspace $(I-P_k)[\mathfrak{X}]$.

where $\xi_k(t, \varepsilon)$ is a solution of equation

$$\frac{d\xi_k}{dt} = \Omega_k(\tau, \varepsilon)\xi_k.$$

The asymptotic nature of the constructed formal solutions is easily proven using a scheme which we utilized earlier (see for example Section 33). In this connection we obtain estimates which are the same as those specified in Theorem 34.4.

In conclusion we note that the consideration of the second-order equation

$$\frac{d^2y}{dt^2} + A(\tau, \varepsilon)\frac{dy}{dt} + B(\tau, \varepsilon)y = 0 \qquad (34.15)$$

in space $\mathfrak{X}$ by means of the conventional approach may be reduced to an equation of the first order in space $\mathfrak{X} \times \mathfrak{X}$.

Indeed, assuming that

$$y = \psi_1, \qquad \frac{dy}{dt} = \psi_2,$$

we have from (34.15),

$$\frac{d\psi_1}{dt} = \psi_2,$$

$$\frac{d\psi_2}{dt} = -B(\tau, \varepsilon)\psi_1 - A(\tau, \varepsilon)\psi_2,$$

$$\frac{d\psi}{dt} = H(\tau, \varepsilon)\psi,$$

where

$$\psi(t, \varepsilon) = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}, \qquad H(\tau, \varepsilon) = \begin{bmatrix} 0 & I \\ -B(\tau, \varepsilon) & -A(\tau, \varepsilon) \end{bmatrix}.$$

Thus the methods presented in the preceding section can be applied to the second-order equation (34.15). /216

Section 35. Appendix

The results which have been presented in Chapter 5 make it possible to solve the problem concerning the asymptotic solution (or partition) of many problems.

To clarify the conditions under which this method is applicable, when specific operators are assigned we must verify the fulfillment of conditions (S_1) and (S_2).

In particular, by using the results of Section 31 we can obtain rather convenient equations for carrying out the asymptotic partitioning of the system of linear differential equations of the n-th order (see Chapter 3) by means of electronic digital computers.

Let us assume that for the system of differential equations of the n-th order:

$$\frac{dx}{dt} = A(\tau, \varepsilon)x$$

the conditions of Theorem 3.1 are satisfied.

To obtain the convenient specified equations for computing the partitioning of the given system we return to the consideration of system (13.7) from Chapter 3:

$$A_0(\tau)U_0(\tau) - U_0(\tau)\mathfrak{A}_0(\tau) = 0, \tag{35.1}$$

$$A_0(\tau)U_s(\tau) - U_s(\tau)\mathfrak{A}_0(\tau) = U_0(\tau)\mathfrak{A}_s(\tau) + B_s(\tau), \tag{35.2}$$

where $B_s(\tau)$ consists of unknown matrices with numbers less than s.

As we know (see Section 13) equation (35.1) is satisfied when

$$U_0(\tau) = V(\tau), \qquad \mathfrak{A}_0(\tau) = W_0(\tau) = V^{-1}(\tau)A_0(\tau)V(\tau), \tag{35.3}$$

where $V(\tau)$ is a matrix which transforms $A_0(\tau)$ into a quasi-diagonal form $W_0(\tau)$. We recall that $V(\tau)$, in accordance with the nature of the matrix $A_0(\tau)$ spectrum, has the structure

$$V(\tau) = [V_1(\tau), V_2(\tau)], \tag{35.4}$$

where $V_k(\tau)$ is a matrix of order (n, r_k) $(k=1, 2; r_1+r_2=n)$ whose columns serve as the base of the subspace which is invariant with respect to $A_0(\tau)$, and additionally they are orthonormalized, i.e., they satisfy the condition

$$V_k^*(\tau)V_k(\tau) = E_{r_k}. \tag{35.5}$$

According to (35.3) and (35.4) and according to the quasi-diagonal structure of the matrix $W_0(\tau)$, the matrix equation (35.2) can be partitioned into the following two equations: /217

$$A_0(\tau)U_{ks}(\tau) - U_{ks}(\tau)W_{k0}(\tau) = V_k(\tau)\mathfrak{A}_{ks}(\tau) + B_{ks}(\tau) \equiv F_{ks}(\tau) \quad (k = 1, 2;\ s = 1, 2, \ldots), \tag{35.6}$$

where

$$B_s(\tau) = \begin{bmatrix} B_{1s}(\tau) \\ B_{2s}(\tau) \end{bmatrix}.$$

In accordance with the results of Section 30 the solution of equation (35.6) may be represented in the form

$$U_{ks}(\tau) = \frac{1}{2\pi i}\int_{\widetilde{\Gamma}_k} [A_0(\tau) - \lambda E]^{-1} F_{ks}(\tau) [W_{k0}(\tau) - \lambda E]^{-1} d\lambda \quad (k = 1, 2;\ s = 1, 2, \ldots), \tag{35.7}$$

if the operator $F_{ks}(\tau)$ satisfies the condition

$$P_k F_{ks}(\tau) = 0. \tag{35.8}$$

In equation (35.7) $\widetilde{\Gamma}_k$ is a smooth contour in the complex plane which contains all the eigenvalues of the matrix $A_0(\tau)$ which do not belong to the k-th group.

Let us assume that the solutions of all equations of form (35.6) with numbers less than some definite number s are already known. Then, from equation (35.6) with the specified number s we can determine the unknown matrices $U_{ks}(\tau)$ and $\mathfrak{A}_{ks}(\tau)$.

Indeed, we find $\mathfrak{A}_{ks}(\tau)$ by using condition (35.8):

$$P_k[V_k(\tau)\mathfrak{A}_{ks}(\tau) + B_{ks}(\tau)] = 0 \tag{35.9}$$

or

$$V_k(\tau)\mathfrak{A}_{ks}(\tau) = -P_k B_{ks}(\tau),$$

from which

$$\mathfrak{A}_{ks}(\tau) = -V_k^*(\tau)P_k B_{ks}(\tau). \tag{35.10}$$

Let us now show how we must carry out the calculations by means of equation (35.7) if $A_0(\tau)$ and $W_{k0}(\tau)$ are square matrices of order n and r_k (k=1, 2) respectively. To make the presentations specific we shall consider the case k=1.

As we know, (ref. 81) the matrix $[W_{10}(\tau)-\lambda E]^{-1}$ is the reciprocal of the matrix $W_{10}(\tau)-\lambda E$ and can be constructed by means of the equation

$$[W_{10}(\tau)-\lambda E]^{-1}=\frac{B}{D_1(\lambda)}, \tag{35.11}$$

where B is a matrix which is allied to $W_{10}(\tau)-\lambda E$, while /21

$$D_1(\lambda)=\det[W_{10}-\lambda E]=\lambda^{r_1}+c_1\lambda^{r_1-1}+\ldots+c_{r_1-1}\lambda+c_{r_1}. \tag{35.12}$$

We recall that a matrix which is allied to some given matrix M (of the n-th order) is one whose form is

$$\begin{bmatrix} M_{11} & M_{21} \ldots M_{n1} \\ M_{12} & M_{22} \ldots M_{n2} \\ \cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot \\ M_{1n} & M_{2n} \ldots M_{nn} \end{bmatrix},$$

where M_{ij} are the algebraic supplements of the element m_{ij} in the determinant of the matrix M.

Since every algebraic supplement in the determinant $\det[W_{10}-\lambda E]$ is a polynomial of the λ degree not greater than r_1-1, the matrix B from equation (35.11) may be represented in the form

$$B=B_0\lambda^{r_1-1}+B_1\lambda^{r_1-2}+\ldots+B_{r_1-2}\lambda+B_{r_1-1}, \tag{35.13}$$

where B_j ($j=0, 1, 2, \ldots, r_1-1$) are matrices which are independent of λ.

In view of the equality

$$(\lambda^{r_1}+c_1\lambda^{r_1-1}+\ldots+c_{r_1-1}\lambda+c_{r_1})E=$$
$$=(B_{r_1-1}+B_{r_1-2}\lambda+\ldots+B_0\lambda^{r_1-1})(W_{10}-\lambda E)$$

we find the following expressions for the matrices $B_j(\tau)$ ($j=0, 1, 2, \ldots, r_1-1$):

$$\begin{aligned} &B_0=-E, \\ &B_j=B_{j-1}W_{10}-c_jE, \qquad j=1,2,\ldots,r_1-1 \\ &(B_{r_1-1}W_{10}=c_{r_1}E). \end{aligned} \tag{35.14}$$

Thus, according to (35.7) and (35.11) we have the following equation for computing the unknown matrix $U_{1s}(\tau)$:

$$U_{1s}(\tau) = \frac{1}{2\pi i} \int_{\tilde{\Gamma}_1} [A_0(\tau) - \lambda E]^{-1} F_{1s}(\tau) \frac{B}{D_1(\lambda)} d\lambda, \tag{35.15}$$

where $\tilde{\Gamma}_1$ is a contour which surrounds the roots of the multinomial $D_2(\lambda)$ (see Chapter 3) and which does not contain the roots of the multinomial $D_1(\lambda)$.

Noting that for the matrix $[A_0(\tau)-\lambda E]^{-1}$ a representation analogous to (35.11) is valid and noting that (see Section 14) /219

$$\frac{1}{D(\lambda)} = \frac{d_1(\lambda)}{D_1(\lambda)} + \frac{d_2(\lambda)}{D_2(\lambda)},$$

we can rewrite (35.15) in the form

$$U_{1s}(\tau) = \frac{1}{2\pi i} \int_{\tilde{\Gamma}_1} d_2(\lambda) [A_0(\tau) - \lambda E]^{-1} F_{1s}(\tau) B d\lambda. \tag{35.16}$$

We note that according to the theory of operational calculus the following equation exists for a bounded operator T:

$$f(T) P_k = -\frac{1}{2\pi i} \int_{\Gamma_k} f(\lambda) (T - \lambda I)^{-1} d\lambda, \tag{35.17}$$

where Γ_k is a smooth contour containing the spectral set σ_k of operator T; $f(\lambda)$ is a rational function which has no poles inside Γ_k.

By applying this equation to the calculation of the integral in (35.16), and taking into account (35.13), (35.14) and (35.18), we find the following expression for the matrix $U_{1s}(\tau)$:

$$\begin{aligned} U_{1s}(\tau) = d_2(A_0) [A_0^{r_1-1} F_{1s}(\tau) + A_0^{r_1-2} F_{1s}(\tau) G_1 + \\ + A_0^{r_1-3} F_{s1}(\tau) G_2 + \ldots + F_{s1}(\tau) G_{r_1-1}] \quad (s = 1, 2, \ldots), \end{aligned} \tag{35.18}$$

where

$$\begin{aligned} G_1 &= -B_1 = W_{10}(\tau) + c_1 E, \\ G_j &= G_{j-1} W_{10}(\tau) + c_j E, \qquad j = 2, 3, \ldots, r_1 - 1 \\ (G_{r_1-1} &W_{10}(\tau) = -c_{r_1} E). \end{aligned} \tag{35.19}$$

In view of condition $P_1K_{1s}(\tau)=0$ and the matrix $U_{1s}(\tau)$ has the same properties, i.e.,

$$P_1U_{1s}(\tau)=0 \qquad (s=1,2,\ldots).$$

By using similar arguments we find the equation for computing the unknown matrix $U_{2s}(\tau)$ if the matrix $\mathfrak{U}_{2s}(\tau)$ is determined from the condition

$$P_2[V_2(\tau)\mathfrak{U}_{2s}(\tau)+B_{2s}(\tau)]=0.$$

The specified equation has the form

$$U_{2s}(\tau)=d_1(A_0)[A_0^{r_2-1}F_{2s}(\tau)+A_0^{r_2-2}F_{2s}(\tau)H_1+\ldots+F_{s2}H_{r_2-1}], \tag{35.18'}$$

where

$$\begin{aligned} H_1&=W_{20}(\tau)+b_1E,\\ H_i&=H_{i-1}(\tau)W_{20}(\tau)+b_iE, \qquad i=2,3,\ldots,r_2-1,\\ &(H_{r_2-1}W_{20}=-b_{r_2}E),\\ D_2(\lambda)&=\lambda^{r_2}+b_1\lambda^{r_2-1}+\ldots+b_{r_2-1}\lambda+b_{r_2}. \end{aligned} \tag{35.19'}$$

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In the case when the eigenvalues of the matrix $A_0(\tau)$ partition into k>2 isolated groups, the corresponding computation formulas may be found in reference 98.

Let us now show, by using a rather trivial example, how the entire computation is carried out for the asymptotic partitioning of a system of linear differential equations.

Let us assume that we are given a system of the third order

$$\begin{aligned} \frac{dx_1}{dt}&=x_2,\\ \frac{dx_2}{dt}&=x_3,\\ \frac{dx_3}{dt}&=(1+0.01t)(2+0.01t)^2x_1-(2+0.01t)(4+0.03t)x_2+\\ &\quad+(5+0.03t)x_3. \end{aligned} \tag{35.20}$$

Assuming that $0.01t=\tau(\varepsilon=0.01)$, we can write the matrix $A(\tau)$ of the corresponding system (35.20) in the following manner:

$$A(\tau)=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ (1+\tau)(2+\tau)^2 & -(2+\tau)(4+3\tau) & 5+5\tau \end{bmatrix}.$$

It is easy to compute that the characteristic polynomial of the matrix $A(\tau)$ has the form

$$D(\lambda)=[\lambda-(1+\tau)][\lambda-(2+\tau)]^2=0, \tag{35.21}$$

and that its roots form two isolated groups:

$$\lambda_1=1+\tau; \qquad \lambda_2=\lambda_3=2+\tau.$$

In accordance with (35.21) we have the equation

$$\frac{1}{D(\lambda)}=\frac{d_1(\lambda)}{\lambda-(1+\tau)}+\frac{d_2(\lambda)}{[\lambda-(2+\tau)]^2},$$

where

$$d_1(\lambda)=1, \qquad d_2(\lambda)=-\lambda+\frac{(2+\tau)^2-1}{1+\tau}. \tag{35.22}$$

Let us determine the projection matrices $P_1(A)$ and $P_2(A)$: /221

$$P_1(A)=[A(\tau)-(2+\tau)E]^2=$$
$$=\begin{bmatrix} (2+\tau)^2 & -2(2+\tau) & 1 \\ (2+\tau)^2(1+\tau) & -2(2+\tau)(1+\tau) & 1+\tau \\ (2+\tau)^2(1+\tau)^2 & -2(2+\tau)(1+\tau)^2 & (1+\tau)^2 \end{bmatrix}, \tag{35.23}$$

$$P_2(A)=\left[-A(\tau)+\frac{(2+\tau)^2-1}{1+\tau}E\right][A(\tau)-(1+\tau)E]=$$
$$=\begin{bmatrix} 1-(2+\tau)^2 & 2(2+\tau) & -1 \\ -(1+\tau)(2+\tau)^2 & 2\tau^2+6\tau+5 & -(1+\tau) \\ -(1+\tau)^2(2+\tau)^2 & 2(2+\tau)(1+\tau)^2 & -\tau(2+\tau) \end{bmatrix}. \tag{35.24}$$

We find the columns of the matrix $V(\tau)=[V(\tau), V_2(\tau)]$, $(V_2^* V_2=E_2)$ from equations $P_1(A)y=0$ and $P_2(A)z=0$:

$$V_1(\tau)=\begin{bmatrix}1\\1+\tau\\(1+\tau)^2\end{bmatrix},\quad V_2(\tau)=\begin{bmatrix}\frac{2}{\alpha} & -\frac{2+\tau}{\beta}\\ \frac{2+\tau}{\alpha} & \frac{2}{\beta}\\ 0 & \frac{(2+\tau)(\tau^2+4\tau+8)}{\beta}\end{bmatrix}, \tag{35.25}$$

where

$$\alpha=\sqrt{\tau^2+4\tau+8},\quad \beta=\sqrt{(2+\tau)^2[1+\alpha^4]+4}.$$

We note that the columns of matrix $V_2(\tau)$ are orthonormalized.

After carrying out the specified preliminary calculations we proceed directly to the partitioning of system (35.20).

According to the nature of the eigenvalues of matrix $A(\tau)$, system (35.20) may be represented asymptotically by means of the transformation

$$x(t,\varepsilon)=U_1(\tau,\varepsilon)\xi_1(t,\varepsilon)+U_2(\tau,\varepsilon)\xi(t,\varepsilon), \tag{35.26}$$

where

$$U_2(\tau,\varepsilon)=\begin{bmatrix}u_{11}(\tau,\varepsilon)\\u_{21}(\tau,\varepsilon)\\u_{31}(\tau,\varepsilon)\end{bmatrix},\quad U_2(\tau,\varepsilon)=\begin{bmatrix}u_{12}(\tau,\varepsilon) & u_{13}(\tau,\varepsilon)\\u_{22}(\tau,\varepsilon) & u_{23}(\tau,\varepsilon)\\u_{32}(\tau,\varepsilon) & u_{33}(\tau,\varepsilon)\end{bmatrix},$$

$\xi_1(t,\varepsilon)$ is a scalar function, $\xi(t,\varepsilon)$ is a two-dimensional vector: /222

$$\xi(t,\varepsilon)=\begin{vmatrix}\xi_2(t,\varepsilon)\\\xi_3(t,\varepsilon)\end{vmatrix},$$

in the form of a single independent equation

$$\frac{d\xi_1}{dt}=a(\tau,\varepsilon)\xi_1 \tag{35.27}$$

and a system of the second order

$$\frac{d\xi_3}{dt}=w_{11}(\tau,\varepsilon)\xi_2+w_{12}(\tau,\varepsilon)\xi_3,\qquad \frac{d\xi_3}{dt}=w_{21}(\tau,\varepsilon)\xi_2+w_{22}(\tau,\varepsilon)\xi_3, \tag{35.28}$$

which corresponds to the matrix

$$W(\tau,\varepsilon)=\begin{bmatrix}w_{11}(\tau,\varepsilon) & w_{12}(\tau,\varepsilon)\\w_{21}(\tau,\varepsilon) & w_{22}(\tau,\varepsilon)\end{bmatrix}.$$

The matrices $U_1(\tau,\varepsilon)$, $U_2(\tau,\varepsilon)$, $W(\tau,\varepsilon)$ and the function $\alpha(\tau,\varepsilon)$, as we know, must satisfy the equations

$$A(\tau)U_1(\tau,\varepsilon)-a(\tau,\varepsilon)U_1(\tau,\varepsilon)=\varepsilon\frac{dU_1}{d\tau}, \tag{35.29}$$

$$AU_2(\tau,\varepsilon)-U_2(\tau,\varepsilon)W(\tau,\varepsilon)=\varepsilon\frac{dU_2}{d\tau}. \tag{35.30}$$

Let us solve equation (35.29) by the method presented above, assuming that for $U_1(\tau,\varepsilon)$ and $\alpha(\tau,\varepsilon)$ the following representation exists:

$$U_1(\tau,\varepsilon)=U_{10}(\tau)+\varepsilon U_{11}(\tau)+\varepsilon^2U_{12}(\tau)+\ldots,$$
$$a(\tau,\varepsilon)=a_0(\tau)+\varepsilon a_1(\tau)+\varepsilon^2a_2(\tau)+\ldots.$$

In view of the results obtained in Section 13 we have

$$U_{10}(\tau)=V_1(\tau)=\begin{bmatrix}1\\1+\tau\\(1+\tau)^2\end{bmatrix},$$

and from equation

$$A(\tau)U_{10}(\tau)-U_{10}(\tau)a_0(\tau)=0$$

we find

$$a_0(\tau)=1+\tau.$$

To determine the vector $U_{11}(\tau)$ and the function $\alpha_1(\tau)$ we obtain the following system from equation (35.29):

$$A(\tau)U_{11}(\tau)-a_0(\tau)U_{11}(\tau)=\frac{dU_{10}}{d\tau}+a_1(\tau)U_{10}(\tau)\equiv F_1(\tau).$$

Utilizing the conditions for the solvability $P_1F_1(\tau)=0$, we find the value of $\alpha_1(\tau)$: /223

$$a_1(\tau)=2.$$

According to the general equation (35.18) the unknown vector $U_{11}(t)$ has the form

$$U_{11}(\tau)=d_2(A)F_1(\tau)=$$
$$=\left[-A(\tau)+\frac{(2+\tau)^2-1}{1+\tau}E\right]\begin{bmatrix}2\\3+2\tau\\2(2+\tau)(1+\tau)\end{bmatrix},$$

i.e.,

$$U_{11}(\tau)=\begin{bmatrix}3\\5+3\tau\\(2+\tau)(4+3\tau)\end{bmatrix}.$$

Thus, if we limit ourselves to the first approximation we obtain the following values for the unknown quantities:

$$U_1^{(1)}(\tau,\varepsilon)=\begin{bmatrix}1.03\\1.05+1.03\tau\\1.08+2.1\tau+1.03\tau^2\end{bmatrix}\equiv$$

$$\equiv\begin{bmatrix}1.03\\1.05+0.0103t\\1.08+0.021t+0.000103t^2\end{bmatrix},$$

$$a^{(1)}(\tau,\varepsilon)=1.02+\tau=1.02+0.01t.$$

Equation (35.30) is solved in the same manner. It is quite understandable that the partitioning of the system of differential equations of a higher order can be achieved only by means of high-speed digital computers. In this case the entire calculation must be carried out at individual fixed points of the interval considered $0\leq t\leq\frac{L}{\varepsilon}$.

Let us show, for example, how the solution of equation (35.30) is constructed in this case.

In accordance with the conventional approach for the asymptotic solution of the given equation we represent $U_2(\tau,\varepsilon)$ and $W(\tau,\varepsilon)$ in the form of power series of ε:

$$U_2(\tau,\varepsilon)=U_{20}(\tau)+\varepsilon U_{21}(\tau)+\varepsilon^2U_{22}(\tau)+\dots,$$
$$W(\tau,\varepsilon)=W_0(\tau)+\varepsilon W_1(\tau)+\varepsilon^2W_2(\tau)+\dots.$$

Then, the matrices $U_{20}(\tau)$ and $W_0(\tau)$ are determined from equation /224

$$A(\tau)U_{20}(\tau)-U_{20}(\tau)W_0(\tau)=0, \tag{35.31}$$

Then the matrices $U_{21}(\tau)$ and $W_1(\tau)$ are determined from

$$A(\tau)U_{21}(\tau)-U_{21}(\tau)W_0(\tau)=\frac{dU_{20}}{d\tau}+U_{20}(\tau)W_1(\tau)\equiv F_2(\tau). \tag{35.32}$$

By selecting m fixed points in the given interval over which the variable $\tau\in[0,L]$ varies, we find the solution of system (35.31) at each of these points.

For example, if we carry out the entire computation according to the scheme specified earlier, at the point $\tau_0=0$ we find

$$U_{20}(\tau_0)=V_2(\tau_0)=\begin{bmatrix}0.707107 & -0,123091\\ 0.707107 & 0.123091\\ 0 & 0.984728\end{bmatrix},$$

$$W_0(\tau)=U_{20}^{*}A(\tau)U_{20}=\begin{bmatrix}0.5 & 0.783351\\ 2.87228 & 3.5\end{bmatrix}.$$

We obtain the corresponding values $U_{20}(\tau_j)$ and $W_0(\tau_j)$ for other fixed τ_j from the interval $[0,L]$.

The resulting values $U_{20}(\tau_j)$ $(j=0, 1, 2, \ldots, m)$ will make it possible to determine the values $\frac{dU_{20}}{d\tau}\Big|_{\tau=\tau_j}$ numerically and consequently to proceed to the solution of system (35.32).

In our case, for $\tau_0=0$ we have

$$\frac{dU_{20}}{d\tau}\Big|_{\tau=\tau_0=0}=\begin{bmatrix}-0.176777 & -0.0587482\\ 0.176777 & -0.120294\\ 0 & 0.0223803\end{bmatrix}.$$

According to the conditions of solvability

$$P_2F_2(\tau)=0,$$

we find

$$W_1(\tau_0)=-U_{20}^{*}(\tau_0)P_2\frac{dU_{20}}{d\tau}\Big|_{\tau=\tau_0=0},$$

i.e.,

$$W_1(\tau_0)=\begin{vmatrix}-2 & +0.506408\\ -1,43614 & 0.25\end{vmatrix}.$$

After this, the solution of equation (35.32)--the matrix $U_{21}(\tau)$--is determined in accordance with (35.18) and (35.19) by means of the equation

$$U_{21}(\tau) = d_1(A)[A(\tau)F_2(\tau) + F_2(\tau)(W_0(\tau) - 2(2+\tau)E)]. \tag{35.33}$$

At the point $\tau_0=0$ the matrix $U_{21}(\tau_0)$ has the form

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$$U_{21}(\tau_0) = |A(\tau_0)F_2(\tau_0) + F_2(\tau_0)(W_0(\tau_0) - 4E)| \equiv$$
$$\equiv \begin{bmatrix} 6.68537 & -1.45472 \\ 6.68537 & -1.45472 \\ 6.68537 & -1.45472 \end{bmatrix}.$$

The calculation for other fixed values $\tau\in[0,L]$ is carried out in the same manner. After the established matrix $U_{21}(\tau)$ is differentiated numerically with respect to τ we can proceed with the determination of the next term in the expansion of matrices $U_2(\tau,\varepsilon)$ and $W(\tau,\varepsilon)$, i.e., to the determination of $U_{22}(\tau)$ and $W_2(\tau)$.

Thus, after completing the calculations, we shall obtain (as a result of the first two steps) the following single equation of first order in place of the system (35.20) of third order:

$$\frac{d\xi_1^{(1)}}{dt} = (1.02 + 0.01t)\xi_1^{(1)} \tag{35.34}$$

and some system of differential equations of the second order. According to the calculations which we have carried out, this partition system has the following form at the instant of time t=0:

$$\frac{d\xi_2^{(1)}}{dt} = 0.048\xi_2^{(1)} + 0.788415\xi_3^{(1)}.$$
$$\frac{d\xi_3^{(1)}}{dt} = 2.85792\xi_2^{(1)} + 3.5025\xi_3^{(1)}. \tag{35.35}$$

Analogous representations are obtained for other fixed values of t. In other words, the coefficients of partition systems are tabulated in carrying out the computations on computers.

The transformation which we have used to achieve the asymptotic partitioning of system (35.20) into two systems (35.34) and (35.35) has the following form at the point t=0:

$$x_1 = 1.03\xi_1^{(1)} + 0.773961\xi_2^{(1)} - 0.136638\xi_3^{(1)},$$
$$x_2 = 1.05\xi_1^{(1)} + 0.773961\xi_2^{(1)} + 0.108544\xi_3^{(1)},$$
$$x_3 = 1.08\xi_1^{(1)} + 0.0668537\xi_2^{(1)} + 0.970181\xi_3^{(1)}.$$

If we had carried out the calculations for other values of t, we would have obtained (in tabular form) the transformation (35.26), which partition system (35.20) into two isolated subsystems (of the first and second order) over the entire interval considered for the variation of t.

CHAPTER 6. ASYMPTOTIC METHODS OF SOLVING LINEAR DIFFERENTIAL EQUATIONS WITH PARTIAL DERIVATIVES

Section 36. Formulation of the Problem

The results of Chapter 5 show that asymptotic methods can be applied to the solution of a series of problems associated with partial differential equations. In this case, as we pointed out in Section 35, it is necessary to investigate the properties of operators which are generated by the specifically formulated problem, i.e., operators assigned by partial differential equations together with the corresponding initial and boundary conditions. If the properties of these operators fall within the framework of conditions (S_1) and (S_2) of Chapter 5, then by using the method described there we can achieve asymptotic partition and sometimes (when the spectrum is simple) find the direct asymptotic solution of the formulated problem.

In this chapter we shall use a single rather simple particular example to show a somewhat different approach to the solution of the above problems. This approach, although it is not completely substantiated theoretically, makes it possible to simplify substantially the computational formulas which have been obtained. In a series of cases, it produces good practical results, as confirmed by comparison of these results with those obtained by other methods and also with experimental data.

Thus, let us consider the following mixed problem.

We are required to find the domain

$$Q = \{0 \leqslant x \leqslant l\} \times \left[0 \leqslant t \leqslant \frac{L}{\varepsilon}\right]$$

the solution of equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \varepsilon a(x, \tau)\, u(x, t) + p(x, \tau) e^{i\theta(t, \varepsilon)}, \tag{36.1}$$

which satisfies the boundary conditions

$$u(0, t) = u(l, t) = 0 \tag{36.2}$$

and the initial conditions

$$u(x, 0) = \varphi_1(x), \quad u_t(x, 0) \equiv \frac{\partial u}{\partial t}(x, 0) = \varphi_2(x). \tag{36.3}$$

(For the sake of simplicity we shall assume that all coefficients in equation (36.1) are real.) /227

As we know, the questions of the existence and uniqueness of solution in

the case of a mixed problem for the hyperbolic equation of the second order and of general form have been well studied (ref. 43). According to the results presented in reference 43, the problem (36.1)-(36.3) has a classical solution* if in the domain Q the coefficients in the free terms of equation (36.1) have continuous derivatives with respect to t and x up to the third order (inclusive), if the functions $\varphi_1(x)$ and $\varphi_2(x)$ can be differentiated a sufficient number of times, $\varphi_2(x)$ can be continuously differentiated four times, and if, in addition, the conditions of compatibility are satisfied (for more details on these conditions see ref. 43).

In the future we shall assume that the above conditions of solvability are satisfied.

It is easy to see that problem (36.1)-(36.3) under the specified assumptions (and even under less stringent ones) may be represented in the form of an operational equation of type (29.1) with the corresponding initial condition. In this case, by using the differential expression $T_u=\frac{\partial^2 u}{\partial x^2}$ and the boundary conditions (36.2), we determine the closed linear operator A, which has the property (S_1), while expression $\alpha(x,\tau)$ in the case of our propositions generates a bounded multiplier operator $B(\tau)$ which has the property (S_2).

Thus, we can see that the results of Chapter 5 are applicable to the asymptotic solution of problem (36.1)-(36.3).

We shall not consider this problem in more detail, but will proceed with the presentation of the approach mentioned above, which leads to a rather simple algorithm for the construction of the asymptotic solution of problem (36.1)-(36.3).

We note finally that in the future, in line with the requirements of the considered asymptotic method, functions $\alpha(x,\tau)p(x,\tau)$ and $\nu(\tau)=\frac{d\theta}{dt}$ from equation (36.1) not only satisfy the conditions of solvability, but also are assumed to be infinitely differentiable with respect to τ on the segment $[0,L]$.**

*The classical solution is interpreted as the function which has continuous derivatives up to the second order in Q and satisfies all the formulated conditions in the conventional sense. In this chapter we shall consider only such solutions.

**As before, the requirement for infinite differentiability may be replaced in practice by the existence of a (sufficient) finite number of partial derivatives with respect to τ.

In this case the following proposition is valid.

The integrals /228

$$I_k(\tau) = \int_0^l \left(\frac{\partial^k a(x,\tau)}{\partial \tau^k}\right)^2 dx, \quad S_k(\tau) = \int_0^l \left(\frac{\partial^k p(x,\tau)}{\partial \tau^k}\right)^2 dx \tag{36.4}$$

for any whole $k \geq 0$ are continuous in τ on the segment $[0, L]$.

Thus we shall seek the solution of equation (34.1) in the form of a series

$$u(x,t) = \sum_{m=1}^{\infty} z_m(t)\, v_m(x), \tag{36.5}$$

where $\nu_m(x)$ is the orthonormalized system of eigenfunctions for the boundary value problem

$$\begin{gathered} \frac{d^2v}{dx^2} + \omega^2 v = 0, \\ v(0) = v(l) = 0. \end{gathered} \tag{36.6}$$

As we know, the eigenvalues ω_n^2 and the corresponding eigenfunctions $\nu_n(x)$ of problem (36.6) have the form

$$\omega_n^2 = \left(n\frac{\pi}{l}\right)^2, \qquad v_n(x) = \sqrt{\frac{2}{l}} \sin \omega_n x. \tag{36.7}$$

Let us substitute expression (36.5) into equation (36.1), assuming that it is possible to carry out a term-by-term differentiation of the series with respect to both variables. Multiplying the resulting equality by $\nu_n(x)$ and integrating both of its parts over x from 0 to 1, we find an infinite system of differential equations, which is satisfied by the unknown functions $z_n(t)$:

$$\begin{gathered} \frac{d^2 z_n}{dt^2} + \omega_n^2 z_n = \varepsilon \sum_{m=1}^{\infty} A_{nm}(\tau) z_m + p_n(\tau) e^{i\theta(t,\varepsilon)} \\ (n = 1, 2, 3, \ldots), \end{gathered} \tag{36.8}$$

where

$$A_{nm}(\tau) = \int_0^l a(\tau, x) v_m(x) v_n(x)\, dx, \qquad (36.9)$$

$$p_n(\tau) = \int_0^l p(\tau, x) v_n(x)\, dx$$

obviously $A_{nm}(\tau) \equiv A_{mn}(\tau)$).

We shall obtain the initial values for system (36.8) if we transform condition (36.3) in an analogous manner, specifically, /229

$$z_n(0) = \int_0^l \varphi_1(x) v_n(x)\, dx = \alpha_n, \qquad (36.10)$$

$$z_n'(0) = \int_0^l \varphi_2(x) v_n(x)\, dx = \beta_n \quad (n = 1, 2, \ldots).$$

In view of our propositions, the coefficients $A_{nm}(\tau)$ and $P_n(\tau)$ are infinitely differentiable with respect to τ on the segment $[0, L]$:

$$\frac{d^k A_{nm}(\tau)}{d\tau^k} = \int_0^l \frac{\partial^k a(\tau, x)}{\partial \tau^k} v_m(x) v_n(x)\, dx, \qquad (36.11)$$

$$\frac{d^k p_n(\tau)}{d\tau^k} = \int_0^l \frac{\partial^k p(\tau, x)}{\partial \tau^k} v_n(x)\, dx \quad (k = 0, 1, 2, \ldots).$$

In this case, according to the Parseval equality, we have

$$\sum_{n=1}^{\infty} \left(\frac{d^k A_{nm}(\tau)}{d\tau^k} \right)^2 = \int_0^l \left(\frac{\partial^k a(\tau, x)}{\partial \tau^k} \right)^2 v_m^2(x)\, dx \leqslant \frac{2}{l} \int_0^l \left(\frac{\partial^k a(\tau, x)}{\partial \tau^k} \right)^2 dx, \qquad (36.12)$$

$$\sum_{n=1}^{\infty} \left[\frac{d^k p_n(\tau)}{d\tau^k} \right]^2 = \int_0^l \left(\frac{\partial^k p(\tau, x)}{\partial \tau^k} \right)^2 dx \quad (k = 0, 1, 2, \ldots).$$

Series (36.12) converge uniformly with respect to $\tau \in [0, L]$ for each $k \geq 0$. This follows directly from the properties of integrals (36.4) and from the well-known Dini theorem (ref. 101) concerning the uniform convergence of functional series with positive terms.

In the next section of this chapter we shall present a sufficiently simple algorithm for constructing the asymptotic solutions of system (36.8) whose

coefficients satisfy the conditions presented above (i.e., they are infinitely differentiable with respect to τ and the series (36.12) converge uniformly).

Section 37. Construction of Formal Solutions

In constructing particular solutions for the system of equations /23

$$\frac{d^2z_n(t)}{dt^2} + \omega_n^2 z_n = \varepsilon \sum_{m=1}^{\infty} A_{nm}(\tau) z_m + p_n(\tau) e^{i\theta(t,\varepsilon)} \quad (n = 1, 2, \ldots), \tag{37.1}$$

we shall distinguish between two cases which are already known:

1) the case of "resonance," when the function $\nu(\tau)$ coincides for certain values $\tau \in [0, L]$, with one number (or with several) from the sequence

$$\omega_n = \frac{\pi}{l} n \quad (n = 1, 2, \ldots);$$

2) the case of "nonresonance," when the function $\nu(\tau)$ does not assume values equal to ω_n (n=1, 2, ...) at any point in $\tau \in [0, L]$.

To make our further presentation convenient we write system (37.1) in the vector-matrix form

$$\frac{d^2Z}{dt^2} + [\Omega - \varepsilon A(\tau)] Z = P(\tau) e^{i\theta(t,\varepsilon)}. \tag{37.2}$$

Here $Z(t, \varepsilon)$ and $P(\tau)$ are vectors of infinite dimensions; Ω is an infinite diagonal matrix whose principal diagonal contains the elements of the numeric sequence ω_n^2 and $A(\tau)$ is an infinite matrix.

We note that in view of the propositions formulated in Section 36, the elements of matrix $A(\tau)$ and of vector $P(\tau)$ can be differentiated indefinitely with respect to τ on the segment $[0, L]$ and series (36.12) converge uniformly.

Let us now consider the construction of a formal particular solution of system (37.2) in the case of "resonance," i.e., when for certain values τ from $[0, L]$ the following exists:

$$\nu(\tau) = \omega_\alpha. \tag{37.3}$$

The algorithm for the construction of the indicated solution is described by the following theorem.

Theorem 6.1. If the elements of matrices $A(\tau)$, Ω and of the vector $P(\tau)$

satisfy the conditions formulated above, then the formal particular solution of system (35.2) in the case of "resonance" may be represented in the form

$$Z_\alpha(t,\varepsilon) = [\Pi_\alpha(\tau,\varepsilon)\xi_\alpha(t,\varepsilon) + f_\alpha(\tau,\varepsilon)]\, e^{i\theta(t,\varepsilon)}, \tag{37.4}$$

where the scalar function $\xi_\alpha(t,\varepsilon)$ satisfies the equation /231

$$\frac{d\xi_\alpha}{dt} = [\lambda_\alpha(\tau,\varepsilon) - i\nu(\tau)]\,\xi_\alpha + b_\alpha(\tau,\varepsilon), \tag{37.5}$$

while the vectors $\Pi_\alpha(\tau,\varepsilon), f_\alpha(\tau,\varepsilon)$ of infinite dimensions and the functions $\lambda_\alpha(\tau,\varepsilon), b_\alpha(\tau,\varepsilon)$ can be expanded formally in the following manner

$$\begin{aligned} \Pi_\alpha(\tau,\varepsilon) &= \sum_{s=0}^{\infty} \varepsilon^s \Pi_\alpha^{(s)}(\tau), \qquad f_\alpha(\tau,\varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s f_\alpha^{(s)}(\tau), \\ \lambda_\alpha(\tau,\varepsilon) &= \sum_{s=0}^{\infty} \varepsilon^s \lambda_\alpha^{(s)}(\tau), \qquad b_\alpha(\tau,\varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s b_\alpha^{(s)}(\tau). \end{aligned} \tag{37.6}$$

Proof. To determine the terms of expansions (37.6), we substitute the expression for vector $Z_\alpha(t,\varepsilon)$ from (37.4) into system (37.3), while taking into account (37.5). Equating separately the coefficients of function $\xi_\alpha(t,\varepsilon)$ and the free terms from both sides of the resulting identity, we find two relationships which can be used to determine the unknown terms of expansions (37.6):

$$\begin{aligned} &\varepsilon^2 \frac{d^2\Pi_\alpha}{d\tau^2} + 2\varepsilon\lambda_\alpha(\tau,\varepsilon)\frac{d\Pi_\alpha}{d\tau} + \varepsilon\frac{d\lambda_\alpha}{d\tau}\Pi_\alpha(\tau,\varepsilon) + \\ &+ [\Omega + \lambda_\alpha^2(\tau,\varepsilon)E]\,\Pi_\alpha(\tau,\varepsilon) = \varepsilon A(\tau)\Pi_\alpha(\tau,\varepsilon), \end{aligned} \tag{37.7}$$

$$\begin{aligned} &\varepsilon^2 \frac{d^2 f_\alpha}{d\tau^2} + 2i\varepsilon\nu(\tau)\frac{df}{d\tau} + i\varepsilon\frac{d\nu}{d\tau}f_\alpha(\tau,\varepsilon) + \\ &+ [\Omega - \nu^2(\tau)E]\,f_\alpha(\tau,\varepsilon) + 2\varepsilon\frac{d\Pi_\alpha}{d\tau}b_\alpha(\tau,\varepsilon) + \\ &+ [\lambda_\alpha(\tau,\varepsilon) + i\nu(\tau)]\,\Pi_\alpha(\tau,\varepsilon)b_\alpha(\tau,\varepsilon) + \varepsilon\Pi_\alpha(\tau,\varepsilon)\frac{db_\alpha}{d\tau}(\tau,\varepsilon) = \\ &= \varepsilon A(\tau) f_\alpha(\tau,\varepsilon) + P(\tau). \end{aligned} \tag{37.8}$$

In identities (37.7) and (37.8), E designates an infinite unit matrix.

1. Let us first consider relationship (37.7). In this relationship we isolate successively the coefficients in front of ε^s (s=0, 1, 2, ...) and obtain recurrence equations for determining the terms $\lambda_\alpha^{(s)}(\tau)$ and $\Pi_\alpha^{(s)}(\tau)$:

$$[\Omega + (\lambda_\alpha^{(0)})^2 E]\,\Pi_\alpha^{(0)}(\tau) = 0,$$

$$[\Omega + (\lambda_\alpha^{(0)}(\tau))^2 E]\,\Pi_\alpha^{(s)}(\tau) = A(\tau)\,\Pi_\alpha^{(s-1)}(\tau) - [2\lambda_\alpha^{(0)}(\tau)\,\lambda_\alpha^{(s)}(\tau) + \tag{37.9}$$

$$+ \sum_{j=1}^{s-1} \lambda_\alpha^{(j)}(\tau)\,\lambda_\alpha^{(s-j)}(\tau)]\,\Pi_\alpha^{(0)}(\tau) - \sum_{k=1}^{s-1}\left[\sum_{j=0}^{s-k} \lambda_\alpha^{(j)}(\tau)\,\lambda_\alpha^{(s-k-j)}(\tau) + \right.$$

$$\left. + \frac{d\lambda_\alpha^{(s-1-k)}(\tau)}{d\tau}\right]\Pi_\alpha^{(k)}(\tau) - 2\sum_{k=1}^{s-1} \lambda_\alpha^{(s-1-k)}(\tau)\,\frac{d\Pi_\alpha^{(k)}(\tau)}{d\tau} - \frac{d^2\Pi_\alpha^{(s-2)}(\tau)}{d\tau^2} -$$

$$- \left[\frac{d\lambda_\alpha^{(s-1)}}{d\tau}\Pi_\alpha^{(0)}(\tau) + 2\lambda_\alpha^{(s-1)}(\tau)\,\frac{d\Pi_\alpha^{(0)}(\tau)}{d\tau}\right] \qquad (s = 1, 2, \ldots). \tag{37.10}$$

In expression (37.10) only those sums have a meaning for which $1 \le k \le s-1$; in addition to this we assume that $\Pi_\alpha^{(-1)} \equiv 0$ in accordance with (37.6).

In accordance with equality (37.9), let us determine the quantities $\lambda_\alpha^{(0)}(\tau)$ and $\Pi_\alpha^{(0)}(\tau)$. To do so we let

$$\Pi_\alpha^{(0)}(\tau) = e_\alpha, \tag{37.11}$$

where e_α is a unit vector of infinite dimensions: the α-th component is equal to unity while the others are equal to zero.

Now relationship (37.9) may be rewritten in the coordinate form as follows:

$$[\omega_n^2 + (\lambda_\alpha^{(0)})^2]\,\delta_{\alpha n} = 0 \qquad (n = 1, 2, \ldots),$$

where $\delta_{\alpha n}$ is the Kronecker symbol.

From this we have

$$\omega_\alpha^2 + (\lambda_\alpha^{(0)})^2 = 0$$

or

$$\lambda_\alpha^{(0)} = \pm i\omega_\alpha.$$

In the future we shall assume that

$$\lambda_\alpha^{(0)} = i\omega_\alpha. \tag{37.12}$$

In accordance with (37.11) and (37.12), we rewrite expression (37.10) in the form

$$[\Omega - \omega_\alpha^2 E]\,\Pi_\alpha^{(s)}(\tau) = F_\alpha^{(s)}(\tau) - 2i\omega_\alpha\lambda_\alpha^{(s)}(\tau)\,e_\alpha, \tag{37.13}$$

where

$$F_\alpha^{(s)}(\tau) = A(\tau)\,\Pi_\alpha^{(s-1)}(\tau) - e_\alpha \sum_{j=1}^{s-1} \lambda_\alpha^{(j)}(\tau)\,\lambda_\alpha^{(s-j)}(\tau) -$$
$$- \sum_{k=1}^{s-1}\left[\sum_{j=0}^{s-k} \lambda_\alpha^{(j)}(\tau)\,\lambda_\alpha^{(s-k-j)}(\tau) + \frac{d}{d\tau}\lambda_\alpha^{(s-1-k)}(\tau)\right]\Pi_\alpha^{(k)}(\tau) - \qquad (37.14)$$
$$- 2\sum_{k=1}^{s-1} \lambda_\alpha^{(s-1-k)}(\tau)\frac{d\Pi_\alpha^{(k)}(\tau)}{d\tau} - \frac{d\lambda_\alpha^{(s-1)}(\tau)}{d\tau}e_\alpha - \frac{d^2\Pi_\alpha^{(s-2)}(\tau)}{d\tau^2}$$
$$(s = 1, 2, \ldots).$$

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If we represent equality (37.13) in the coordinate form

$$(\omega_n^2 - \omega_\alpha^2)\,\Pi_{n\alpha}^{(s)}(\tau) = F_{n\alpha}^{(s)}(\tau) - 2i\omega_\alpha\lambda_\alpha^{(s)}(\tau)\,\delta_{\alpha n}, \qquad (37.15)$$

we can successively determine $\lambda_\alpha^{(s)}(\tau)$ and the components of the vector $\Pi_{\alpha n}^{(s)}(\tau)$ for any number $s \geq 1$. Indeed, when $n \neq \alpha$ we have, from equality (37.15),

$$\Pi_{n\alpha}^{(s)}(\tau) = \frac{F_{n\alpha}^{(s)}(\tau)}{\omega_n^2 - \omega_\alpha^2}, \qquad (37.16)$$

or

$$\Pi_{n\alpha}^{(s)}(\tau) = \frac{1}{\omega_n^2 - \omega_\alpha^2}\left[\sum_{j=1}^{\infty} A_{nj}(\tau)\,\Pi_{j\alpha}^{(s-1)}(\tau) - \right.$$
$$- \sum_{k=1}^{s-1}\left[\sum_{j=0}^{s-k} \lambda_\alpha^{(j)}(\tau)\,\lambda_\alpha^{(s-k-j)}(\tau) + \frac{d\lambda_\alpha^{(s-1-k)}(\tau)}{d\tau}\right]\Pi_{n\alpha}^{(k)}(\tau) - \qquad (37.17)$$
$$\left. - 2\sum_{k=1}^{s-1} \lambda_\alpha^{(s-1-k)}(\tau)\frac{d\Pi_{n\alpha}^{(k)}(\tau)}{d\tau} - \frac{d^2\Pi_{n\alpha}^{(s-2)}(\tau)}{d\tau^2}\right]$$
$$(n = 1, 2, \ldots; \quad n \neq \alpha; \quad s = 1, 2, \ldots).$$

In the case when $n=\alpha$, it follows from equality (37.15) that

$$F_{\alpha\alpha}^{(s)}(\tau) - 2i\omega_\alpha\lambda_\alpha^{(s)}(\tau) = 0 \qquad (s = 1, 2, \ldots), \qquad (37.18)$$

and the function $\Pi_{\alpha\alpha}^{(s)}(\tau)$ may be arbitrary, and as a result we can assume (for the sake of simplicity) that

$$\Pi_{\alpha\alpha}^{(s)}(\tau) \equiv 0 \qquad (s = 1, 2, \ldots). \qquad (37.19)$$

Now, according to equation (37.14) and equality (37.18), we find

$$\lambda_\alpha^{(s)}(\tau)=\frac{1}{2i\omega_\alpha}\left[\sum_{j=1}^{\infty}A_{\alpha j}(\tau)\Pi_{j\alpha}^{(s-1)}(\tau)-\right.$$
$$\left.-\sum_{j=1}^{s-1}\lambda_\alpha^{(j)}(\tau)\lambda_\alpha^{(s-j)}(\tau)-\frac{d\lambda_\alpha^{(s-1)}(\tau)}{d\tau}\right]\quad(s\geqslant 1). \tag{37.20}$$

However, expressions (37.17) and (37.20) may be looked upon as formulas which determine the terms of the expansions for the unknown functions $\lambda_\alpha(\tau,\varepsilon)$ and $\Pi_\alpha(\tau,\varepsilon)$ only when the uniform convergence of series which play a role in (37.17) and (37.20) is proven and when the differentiability of the respective elements is established. /234

Let us investigate equations (37.17) and (37.20) from this point of view.

We note that $\Pi_\alpha^{(0)}=e_\alpha$ and $\lambda_\alpha^{(0)}=i\omega_\alpha$ are infinitely differentiable.

In the case of s=1, we have, in accordance with (37.17) and (37.20),

$$\Pi_{n\alpha}^{(1)}(\tau)=\frac{A_{n\alpha}(\tau)}{\omega_n^2-\omega_\alpha^2}\ (n=1,2,\ldots,n\neq\alpha), \tag{37.21}$$

$$\Pi_{\alpha\alpha}^{(1)}\equiv 0,$$

$$\lambda_\alpha^{(1)}(\tau)=\frac{A_{\alpha\alpha}(\tau)}{2i\omega_\alpha}. \tag{37.22}$$

It follows from this that $\Pi_{n\alpha}^{(1)}(\tau)$ and $\lambda_\alpha^{(1)}(\tau)$ are completely determined by equations (37.17) and (37.20) and have the same number of derivatives with respect to τ as the elements of the matrix $A(\tau)$. In addition to this, it is easy to show that the series

$$\sum_{n=1}^{\infty}\left|\frac{d^k\Pi_{n\alpha}^{(1)}(\tau)}{d\tau^k}\right|^2<\infty \tag{37.23}$$

for any whole, $k\geq 0$ converges uniformly with respect to $\tau\in[0,L]$.

Indeed, according to (37.21), for $k\geq 0$, we have

$$\sum_{n=1}^{\infty}\left|\frac{d^k\Pi_{n\alpha}^{(1)}(\tau)}{d\tau^k}\right|^2=\sum_{n=1}^{\alpha-1}(\omega_n^2-\omega_\alpha^2)^2\left|\frac{d^kA_{n\alpha}}{d\tau^k}\right|^2+\sum_{n=\alpha+1}^{\infty}\frac{1}{(\omega_n^2-\omega_\alpha^2)^2}\left|\frac{d^kA_{n\alpha}}{d\tau^k}\right|^2, \qquad (37.24)$$

and the series in the right side of equality (37.24) converges uniformly in view of our propositions (compare with the corresponding series (36.12)) and in view of the Abel's condition for uniform convergence (ref. 101).

Let us now assume that equations (37.17) and (37.20) are meaningful for all numbers s which are less than some natural number m($s\leq m-1$) and that the functions $\Pi_{n\alpha}^{(s)}(\tau),\lambda_{\alpha}^{(s)}(\tau)$ can be differentiated with respect to τ (obviously the number of derivatives for $\lambda_{\alpha}^{(s)}(\tau)$ and $\Pi_{\alpha n}^{(s)}(\tau)$ will be (s-1) times less than for the elements of the matrix $A(\tau)$ if the latter are not infinitely differentiable.

In addition to this we shall assume that the series

$$\sum_{n=1}^{\infty}\left|\frac{d^k\Pi_{n\alpha}^{(s)}(\tau)}{d\tau^k}\right|^2<\infty\ (0\leqslant s\leqslant m-1,\ k\geqslant 0), \qquad (37.25)$$

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which are constructed for any of the existing arbitrary functions $\Pi_{n\alpha}^{(s)}(\tau)$ converge uniformly with respect to τ.

It is easy to see that under the assumptions adopted the functions $\Pi_{n\alpha}^{(m)}(\tau)$ and $\lambda_{\alpha}^{(m)}(\tau)$ are entirely determined by equations (37.17) and (37.20), are differentiable with respect to τ, and the series

$$\sum_{n=1}^{\infty}\left|\frac{d^k\Pi_{n\alpha}^{(m)}(\tau)}{d\tau^k}\right|^2 (k=0,1,2,\ldots) \qquad (37.26)$$

converges uniformly on the segment [0,L].

Indeed, the infinite series in expressions (37.17) and (37.20) converge when s=m and do this uniformly because according to the Cauchy-Bunyakovskiy inequality, the following estimate exists:

$$\sum_{j=1}^{\infty}|A_{nj}(\tau)\Pi_{j\alpha}^{(m-1)}(\alpha)|\leqslant\sqrt{\sum_{j=1}^{\infty}|A_{nj}(\tau)|^2}\ \sqrt{\sum_{j=1}^{\infty}|\Pi_{j\alpha}^{(m-1)}(\tau)|^2}$$
$$(n=1,2,\ldots).$$

In the same way we can become convinced that the following series converge uniformly:

$$\sum_{j=1}^{\infty}\left|\frac{d}{d\tau}\left[A_{nj}(\tau)\,\Pi_{j\alpha}^{(m-1)}(\tau)\right]\right| \leqslant \sum_{j=1}^{\infty}\left|\frac{dA_{nj}}{d\tau}\,\Pi_{j\alpha}^{(m-1)}(\tau)\right| + \sum_{j=1}^{\infty}\left|A_{nj}(\tau)\,\frac{d\Pi_{j\alpha}^{(m-1)}(\tau)}{d\tau}\right|,$$

i.e., the function $\Pi_{n\alpha}^{(m)}(\tau)$ and $\lambda_{\alpha}^{(m)}(\tau)$ are differentiable with respect to τ (see equations (37.17) and (37.20)).

Furthermore, in view of our propositions and in view of equations (37.17) and (37.20), the functions $\lambda_{\alpha}^{(m)}(\tau)$ and $\Pi_{n\alpha}^{(m)}(\tau)$ have only one derivative which is less than $\lambda_{\alpha}^{(m-1)}(\tau)$ and $\Pi_{n\alpha}^{(m-1)}(\tau)$.

In order to become convinced that series (37.26) converges uniformly, it is sufficient to substitute the value of $\Pi_{n\alpha}^{(m)}(\tau)$ from expression (37.17) into series (37.26) and to carry out the corresponding evaluation of the resulting expression. Without writing in detail (due to awkwardness) the general term of series (37.26), we shall evaluate only several typical expressions. For example, when investi- /236 gating the convergence of the series

$$\sum_{n=1}^{\infty}\frac{1}{(\omega_n^2-\omega_\alpha^2)^2}\left|\frac{d^k}{d\tau^k}\sum_{j=1}^{\infty}A_{nj}(\tau)\,\Pi_{j\alpha}^{(m-1)}(\tau)\right|^2$$

(see equations (37.17) and (37.26)), we encounter series of the form

$$\sum \equiv \sum_{n=1}^{\infty}\frac{1}{(\omega_n^2-\omega_\alpha^2)^2}\left|\sum_{j=1}^{\infty}\frac{d^{k-l}A_{nj}}{d\tau^{k-l}}\,\frac{d^l\Pi_{j\alpha}^{(m-1)}(\tau)}{d\tau^l}\right|^2, \qquad (37.27)$$

where $0 \leq l \leq k$.

Using the Cauchy-Bunyakovskiy inequality, we find

$$\sum \leqslant \sum_{n=1}^{\infty}\left[\frac{1}{(\omega_n^2-\omega_\alpha^2)^2}\sum_{j=1}^{\infty}\left|\frac{d^{k-l}A_{nj}}{d\tau^{k-l}}\right|\right]\sum_{j=1}^{\infty}\left|\frac{d^l\Pi_{j\alpha}^{(m-1)}(\tau)}{d\tau^l}\right|^2,$$

'rom which (in view of (36.12) and (36.25)) the uniform convergence of series 37.27) is obvious.

Furthermore, the question concerning the uniform convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{(\omega_n^2 - \omega_\alpha^2)^2} \left| \frac{d^k}{d\tau^k} \sum_{i=1}^{m-1} \beta_i(\tau) \Pi_{n\alpha}^{(i)}(\tau) \right|^2,$$

which is a composite part of expression (37.26), where

$$\beta_i(\tau) = -\sum_{j=0}^{m-i} [\lambda_\alpha^{(j)}(\tau, \lambda_\alpha^{(m-i-j)}(\tau) + \lambda_\alpha^{(m-1-i)}(\tau)],$$

is reduced to the investigation of the convergence of series having the form

$$\sum_{n=1}^{\infty} \frac{1}{(\omega_n^2 - \omega_\alpha^2)^2} \left| \frac{d^l \Pi_{n\alpha}^{(i)}(\tau)}{d\tau^l} \right|^2 \quad (0 \leqslant l \leqslant k, \quad 1 \leqslant i \leqslant m-1),$$

which, according to our propositions, converge uniformly with respect to $\tau \in [0, L]$.

Thus, we can become convinced that series (37.26) does indeed converge uniformly if proposition (37.25) is satisfied.

Thus, the recurrence equations (37.17) and (37.20) make it possible to determine the terms of expansions (37.6) for the functions $\lambda_\alpha(\tau)$ and the vector $\Pi_\alpha(\varepsilon, \tau)$ when such terms have any number s and the functions are differentiable with respect to τ by (s-1) times less than the elements of the matrix $A(\tau)$, if the latter has a finite number of derivatives.

2. Let us now determine the terms of expansion (37.6) of function $b_\alpha(\tau, \varepsilon)$ and vector $f_\alpha(\tau, \varepsilon)$. For this purpose we make use of identity (37.8). If in /237 both sides of this identity we equate its coefficients for the same powers of parameter ε, we then shall obtain recurrence relationships which can be used to find the specified terms of the expansion

$$[\Omega - \nu^2(\tau) E] f_\alpha^{(0)}(\tau) + [\lambda_\alpha^{(0)}(\tau) + i\nu(\tau)] \Pi_\alpha^{(0)}(\tau) b_\alpha^{(0)}(\tau) = P(\tau), \tag{37.28}$$

$$[\Omega - \nu^2(\tau) E] f_\alpha^{(s)}(\tau) + [\lambda_\alpha^{(0)} + i\nu(\tau)] \Pi_\alpha^{(0)}(\tau) b_\alpha^{(s)}(\tau) =$$

$$= \left[A(\tau) - i \frac{d\nu}{d\tau} E \right] f_\alpha^{(s-1)}(\tau) - \sum_{j=1}^{s} \left[i\nu(\tau) \Pi_\alpha^{(j)}(\tau) + \right.$$

$$+\sum_{k=0}^{l}\lambda_{\alpha}^{(k)}(\tau)\,\Pi_{\alpha}^{(l-k)}(\tau)\Big]b_{\alpha}^{(s-l)}(\tau)-\sum_{j=0}^{s-1}\Pi_{\alpha}^{(j)}(\tau)\frac{db_{\alpha}^{(s-1-j)}(\tau)}{d\tau}-$$

$$-2\sum_{j=0}^{s-1}\frac{d\Pi_{\alpha}^{(j)}(\tau)}{d\tau}b_{\alpha}^{(s-1-j)}(\tau)-\frac{d^2f^{(s-2)}(\tau)}{d\tau^2}-2i\nu(\tau)\frac{df_{\alpha}^{(s-1)}(\tau)}{d\tau}. \tag{37.29}$$

By using the results of Subsection 1 of this section we can represent equality (37.28) in a coordinate form

$$[\omega_n^2-\nu^2(\tau)]\,f_{n\alpha}^{(0)}(\tau)+i\,[\omega_{\alpha}+\nu(\tau)]\,b_{\alpha}^{(0)}(\tau)\,\delta_{n\alpha}=p_n(\tau), \tag{37.30}$$

from which we easily find $f_{n\alpha}^{(0)}(\tau)$ and $b_{\alpha}^{(0)}(\tau)$.

Let $n=\alpha$. Then

$$[\omega_{\alpha}^2-\nu^2(\tau)]\,f_{\alpha\alpha}^{(0)}(\tau)+i\,[\omega_{\alpha}+\nu(\tau)]\,b_{\alpha}^{(0)}(\tau)=p_{\alpha}(\tau). \tag{37.31}$$

Since in the case of "resonance" for certain values of $\tau\in[0,L]$ the values $\nu(\tau)$ and ω_{α} coincide, we shall select $b_{\alpha}^{(0)}(\tau)$ in accordance with (37.31) such that the following equality is satisfied on the entire segment $[0,L]$:

$$i\,[\omega_{\alpha}+\nu(\tau)]\;b_{\alpha}^{(0)}(\tau)=p_{\alpha}(\tau), \tag{37.32}$$

i.e.,

$$b_{\alpha}^{(0)}(\tau)=\frac{p_{\alpha}(\tau)}{i\,[\omega_{\alpha}+\nu(\tau)]}. \tag{37.33}$$

It is obvious that with selection of $b_{\alpha}^{(0)}(\tau)$ we must let (in accordance with (37.31) and (37.32))

$$f_{\alpha\alpha}^{(0)}(\tau)\equiv 0. \tag{37.34}$$

When $n\neq\alpha$ we obtain the following from equality (37.30):

$$f_{n\alpha}^{(0)}(\tau)=\frac{p_n(\tau)}{\omega_n^2-\nu^2(\tau)},\qquad n=1,2,\ldots,\ n\neq\alpha. \tag{37.35}$$

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In the same way we can successively determine $f_{n\alpha}^{(s)}(\tau)$ and $b_{\alpha}^{(s)}(\tau)$ for any natural $s\geq 1$ from relationship (37.29).

We rewrite (37.29) in the coordinate form taking into account the results of Subsection n.1:

$$[\omega_n^2 - \nu^2(\tau)]\, f_{n\alpha}^{(s)}(\tau) + i\,[\omega_\alpha + \nu(\tau)]\, b_\alpha^{(s)}(\tau)\delta_{n\alpha} = T_{n\alpha}^{(s)}(\tau) \qquad (37.36)$$
$$(n = 1, 2, \ldots;\ s = 1, 2, \ldots),$$

where

$$T_{n\alpha}^{(s)}(\tau) = \sum_{j=1}^{\infty} A_{nj}(\tau)\, f_{j\alpha}^{(s-1)}(\tau) - i\frac{d\nu}{d\tau} f_{n\alpha}^{(s-1)}(\tau) - \sum_{l=1}^{s}\Bigg[i\nu(\tau)\Pi_{n\alpha}^{(l)}(\tau) +$$
$$+ \sum_{k=0}^{l} \lambda_\alpha^{(k)}(\tau)\,\Pi_{n\alpha}^{(l-k)}(\tau)\Bigg] b_\alpha^{(s-l)}(\tau) - \sum_{l=0}^{s-1} \Pi_{n\alpha}^{(l)}(\tau)\frac{db_\alpha^{(s-1-l)}(\tau)}{d\tau} - \qquad (37.37)$$
$$- 2\sum_{l=0}^{s-1} \frac{d\Pi_{n\alpha}^{(l)}(\tau)}{d\tau}\, b_\alpha^{(s-1-l)}(\tau) - \frac{d^2 f_{n\alpha}^{(s-2)}(\tau)}{d\tau^2} - 2i\nu(\tau)\frac{df_{n\alpha}^{(s-1)}(\tau)}{d\tau}.$$

When $n=\alpha$, relationship (37.36) has the form

(37.38)

$$[\omega_\alpha^2 - \nu^2(\tau)]\, f_{\alpha\alpha}^{(s)}(\tau) + i\,[\omega_\alpha + \nu(\tau)]\, b_\alpha^{(s)}(\tau) = T_{\alpha\alpha}^{(s)}(\tau),$$

while

$$T_{\alpha\alpha}^{(s)}(\tau) = \sum_{j=1}^{\infty} A_{\alpha j}(\tau)\, f_{j\alpha}^{(s-1)}(\tau) - i\frac{d\nu}{d\tau} f_{\alpha\alpha}^{(s-1)}(\tau) -$$
$$- \sum_{l=1}^{s} \lambda_\alpha^{(l)}(\tau)\, b_\alpha^{(s-l)}(\tau) - \frac{db_\alpha^{(s-1)}(\tau)}{d\tau} - \frac{d^2 f_{\alpha\alpha}^{(s-2)}(\tau)}{d\tau^2} - 2i\nu(\tau)\frac{df_{\alpha\alpha}^{(s-1)}(\tau)}{d\tau}. \qquad (37.39)$$

Carrying out the same arguments as in the case s=0, we select $b_\alpha^{(s)}(\tau)$ in equality (37.38) in such a way that for any $\tau\in[0,L]$ the following equality is satisfied:

$$i\,[\omega_\alpha + \nu(\tau)]\, b_\alpha^{(s)}(\tau) = T_{\alpha\alpha}^{(s)}(\tau),$$

in other words,

$$b_\alpha^{(s)}(\tau) = \frac{T_{\alpha\alpha}^{(s)}(\tau)}{i\,[\omega_\alpha + \nu(\tau)]} \qquad (s = 1, 2, \ldots). \qquad (37.40)$$

As a result of this selection of $b_\alpha^{(s)}(\tau)$, we must let /239

$$f_{\alpha\alpha}^{(s)}(\tau) \equiv 0 \qquad (s = 1, 2, \ldots). \qquad (37.41)$$

in equality (37.38).

Returning to equation (37.40), we can represent it (by taking into account (37.39) and (37.41)) in the expanded form

$$b_\alpha^{(s)}(\tau)=\frac{1}{i[\omega_\alpha+\nu(\tau)]}\left[\sum_{j=1}^{\infty}A_{\alpha j}(\tau)f_{j\alpha}^{(s-1)}(\tau)-\sum_{j=1}^{s}\lambda_\alpha^{(j)}(\tau)b_\alpha^{(s-j)}(\tau)-\frac{db_\alpha^{(s-1)}(\tau)}{d\tau}\right] \qquad (37.42)$$

$$(s=1,2,\ldots).$$

Let $n\neq\alpha$. Then, according to (37.36) we have

$$[\omega_n^2-\nu^2(\tau)]f_{n\alpha}^{(s)}(\tau)=T_{n\alpha}^{(s)}(\tau),$$

whence it follows that

$$f_{n\alpha}^{(s)}(\tau)=\frac{T_{n\alpha}^{(s)}(\tau)}{\omega_n^2-\nu^2(\tau)} \quad (n=1,2,\ldots;\quad n\neq\alpha,\quad s=1,2,\ldots). \qquad (37.43)$$

As in the case of Subsection 1, we must become convinced that equations (37.42) and (37.43) are meaningful, i.e., that the infinite series in these equations converge and permit a term-by-term differentiation with respect to τ.

It follows from equations (37.33) and (37.35) that the functions $b_\alpha^{(0)}(\tau)$ and $f_{n\alpha}^{(0)}(\tau)$ have the same number of derivatives as the components of vector $P(\tau)$ and the function $\nu(\tau)$. In addition to this, it is easy to show that the series

$$\sum_{n=1}^{\infty}\left|\frac{d^m}{d\tau^m}f_{n\alpha}^{(0)}(\tau)\right|^2 \qquad (m=0,1,2,\ldots) \qquad (37.44)$$

converges uniformly.

Indeed, by writing out the general term of series (37.44) in detail, we have, in accordance with (37.35),

$$\sum_{n=1}^{\infty}\left|\frac{d^m}{d\tau^m}f_{n\alpha}^{(0)}(\tau)\right|^2=\sum_{n=1}^{\alpha-1}\left|\frac{d^mf_{n\alpha}^{(0)}(\tau)}{d\tau^m}\right|^2+\sum_{n=\alpha+1}^{\infty}\left|\sum_{j=0}^{m}C_m^j\times\right. \qquad (37.45)$$

$$\left.\times\left[\frac{d^{m-j}}{d\tau^{m-j}}\cdot\frac{1}{\omega_n^2-\nu^2(\tau)}\right]\frac{d^jp_n(\tau)}{d\tau^j}\right|^2 \qquad \left(C_m^j=\frac{m!}{j!(m-j)!}\right).$$

Thus, the question concerning the convergence of series (37.44) is reduced to /240

the investigation of the convergence of series

$$\sum_{n=\alpha+1}^{\infty}\left|\sum_{j=0}^{m}C_m^j\left[\frac{d^{m-j}}{d\tau^{m-j}}\cdot\frac{1}{\omega_n^2-\nu^2(\tau)}\right]\frac{d^j p_n(\tau)}{d\tau^j}\right|^2. \tag{37.46}$$

Let us first consider the expression of the k-th derivative of the function $\frac{1}{\omega_n^2-\nu^2(\tau)}$. Obviously, the following relationship exists:

$$\frac{d^k}{d\tau^k}\left[\frac{1}{\omega_n^2-\nu^2(\tau)}\right]=\sum_{j=1}^{k}\frac{f_j(\nu(\tau),\nu'(\tau),\ldots,\nu^{(k)}(\tau))}{[\omega_n^2-\nu^2(\tau)]^{j+1}}, \tag{37.47}$$

where $f_j[\nu(\tau),\nu'(\tau),\ldots,\nu^{(k)}(\tau)]$ is a multinomial of the function $\nu(\tau)$ and of its derivatives with respect to τ up to the k-th order inclusive.

The polynomial $f_j(\nu,\nu'\ldots,\nu^{(k)})$ does not depend on ω_n, and because the function $\nu(\tau)$ is infinitely differentiable, it is bounded with respect to τ on the segment $[0,L]$.

$$|f_j(\nu,\nu',\ldots,\nu^{(k)})|<M_j=\text{const}. \tag{37.48}$$

In accordance with (37.47) and (37.48), the following estimate exists for the series (37.46):

$$\sum_{n=\alpha+1}^{\infty}\left|\sum_{j=0}^{m}C_m^j\left[\frac{d^{m-j}}{d\tau^{m-j}}\cdot\frac{1}{\omega_n^2-\nu^2(\tau)}\right]\frac{d^j p_n}{d\tau^j}\right|^2\leqslant$$

$$\leqslant\sum_{n=\alpha+1}^{\infty}\left\{\sum_{j=0}^{m}(C_m^j)^2(m-j)\sum_{k=1}^{m-j}\frac{M_k^2}{[\omega_n^2-\nu^2(\tau)]^{2k+2}}\sum_{j=0}^{m}\left(\frac{d^j p_n(\tau)}{d\tau^j}\right)^2\right\}\leqslant$$

$$\leqslant\left[\sum_{j=0}^{m}(C_m^j)^2(m-j)\sum_{k=1}^{m-j}\frac{M_k^2}{[\omega_{\alpha+1}^2-\nu^2(\tau)]^{2k+2}}\right]\sum_{j=0}^{m}\sum_{n=\alpha+1}^{\infty}\left(\frac{d^j p_n}{d\tau^j}\right)^2.$$

From this, in view of our propositions, the uniform convergence of series (37.46), and consequently of series (37.44) is obvious.

By carrying out analogous considerations we can show that equations (37.42)

and (37.43) have a meaning for any natural $s \geq 1$ and the functions $b_\alpha^{(s)}(\tau)$ and $f_{n\alpha}^{(s)}(\tau)$ which they determine are differentiable with respect to τ, and such differentiability is only s times less than that of functions $b_\alpha^{(0)}$ and $f_{n\alpha}^{(0)}(\tau)$ if the latter are not infinitely differentiable.

In addition to this, the series /241

$$\sum_{n=1}^{\infty} \left| \frac{d^m}{d\tau^m} f_{n\alpha}^{(s)}(\tau) \right|^2 \qquad (m \geqslant 0)$$

converge uniformly with respect to τ for any $s \geq 1$.

(We shall not consider this question in detail, since the calculations are cumbersome and are of the same type as those previously considered.)

Thus, in Subsections 1 and 2 of the present section we have presented an algorithm which can be used to construct a particular solution of the infinite system (37.1) in the case of "resonance." This concludes the proof of Theorem 5.1.

Note 1. In the case of a more complex "resonance," when the function $\nu(\tau)$ for certain values of $\tau \in [0, L]$ coincides with several numbers ω_k, for example, with the numbers $\omega_1, \omega_2, \ldots, \omega_r$ and cannot coincide with other terms of the sequence ω_n $(n = r+1, r+2, \ldots)$ for any values of τ, the construction of the particular solution of system (37.1) is carried out in the same way as described in Subsections 1 and 2. Specifically, the particular solution $Z(t, \varepsilon)$ of the nonhomogeneous system (37.2) is sought in the form

$$Z(t, \varepsilon) = \left[\sum_{k=1}^{r} \Pi_k(\tau, \varepsilon) \xi_k(t, \varepsilon) + f(\tau, \varepsilon) \right] e^{i\theta(t, \varepsilon)},$$

where the scalar functions $\xi_k(t, \varepsilon)$ are solutions of equations

$$\frac{d\xi_k}{dt} = [\lambda_k(\tau, \varepsilon) - i\nu(\tau)]\, \xi_k + b_k(\tau, \varepsilon) \qquad (k = 1, 2, \ldots, r).$$

The expansion terms of vectors $\Pi_k(\tau, \varepsilon), f(\tau, \varepsilon)$ and functions $\lambda_k(\tau, \varepsilon)$ in this case are determined by equations which are analogous to those presented in Subsections 1 and 2.

Note 2. In the case of an absence of "resonance," the particular solution of system (37.2) is sought, according to the rule, in the form

$$Z(t, \varepsilon) = f(\tau, \varepsilon) e^{i\theta(t,\varepsilon)},$$

where $f(\tau, \varepsilon)$ is the unknown vector of infinite dimensions.

In this case, due to the absence of functions $\xi_k(t, \varepsilon)$, the derivation of approximate solutions for the initial system is reduced merely to the solution of algebraic equations and to differentiation.

Note 3. If in the mixed problem considered a homogeneous equation corresponding to (36.1) is assigned, then the unknown functions $z_n(t, \varepsilon)$ (n=1, 2, ...) (see equation (36.5)) are determined from the homogeneous system

$$\frac{d^2 z_n}{dt^2} + \omega_n^2 z_n = \varepsilon \sum_{m=1}^{\infty} A_{nm}(\tau) z_m \qquad (n = 1, 2, \ldots).$$

In order to find the particular solution of this system, we also apply the method which has just been described. In this case, the particular solution corresponding to the eigenvalue ω_k is sought in the form /242

$$Z_k(t, \varepsilon) = \Pi_k(\tau, \varepsilon) \xi_k(t, \varepsilon),$$

where $\Pi_k(\tau, \varepsilon)$ and $\xi_k(t, \varepsilon)$ are the same as above but the function $\xi_k(t, \varepsilon)$ satisfies the homogeneous equation of the form

$$\frac{d\xi_k}{dt} = \lambda_k(\tau, \varepsilon) \xi_k \qquad (k = 1, 2, \ldots).$$

Section 38. Proof of Asymptotic Convergence

The algorithm described in Section 37 makes it possible to determine any term of the expansion in equations (37.6). Usually in practice, because of the cumbersome calculations, it becomes necessary to limit ourselves to the construction of the first m terms of the above expansions. For this reason we introduce the so-called m-th approximations with unknown solutions of system (37.2). Let us determine the m-th approximation for the case of "resonance" considered in Section 37, with the aid of expression

$$Z_\alpha^{(m)}(t, \varepsilon) = \left[\sum_{s=0}^{m} \varepsilon^s \Pi_\alpha^{(s)}(\tau) \xi_\alpha^{(m)}(t, \varepsilon) + \sum_{s=0}^{m} \varepsilon^s f^{(s)}(\tau) \right] e^{i\theta(t,\varepsilon)}, \qquad (38.1)$$

where the function $\xi_\alpha^{(m)}(t, \varepsilon)$ is the solution of equation

$$\frac{d\xi_\alpha^{(m)}}{dt} = \left[\sum_{s=0}^{m} \varepsilon^s \lambda_\alpha^{(s)}(\tau) - i\nu(\tau)\right] \xi_\alpha^{(m)} + \sum_{s=0}^{m} \varepsilon^s b_\alpha^{(s)}(\tau). \tag{38.2}$$

Let us assume that $Z(t,\varepsilon)$ represents the exact solution of system (37.2), which satisfies the same initial conditions for t=0 as $Z_\alpha^{(m)}(t,\varepsilon)$. Then, by using the same methods as in the preceding chapters, (particularly in the second chapter), we can show that the approximate solution $Z_\alpha^{(m)}(t,\varepsilon)$ converges asymptotically to the exact solution $Z(t,\varepsilon)$.

Without considering the details of the proof, we shall indicate only its basic stages.

First of all we prove the lemma.

Lemma 6.1. Let us assume that the conditions of Theorem 6.1 are satisfied. Then the m-th approximation $Z_\alpha^{(m)}(t,\varepsilon)$ satisfies the equation

$$\frac{d^2 Z_\alpha^{(m)}(t,\varepsilon)}{dt^2} + [\Omega - \varepsilon A(\tau)]\, Z_\alpha^{(m)}(t,\varepsilon) = P(\tau) e^{i\theta(t,\varepsilon)} + \varepsilon^m R(\tau,\varepsilon), \tag{38.3}$$

where $R(\tau,\varepsilon)$ is a vector function which is uniformly bounded on the segment [0,L].* /243

The proof of Lemma 6.1 is carried out in exactly the same manner as the proof of Lemma 2.1., specifically, by substituting expression (38.1) into equation (37.2) (taking into account the method of constructing $Z_\alpha^{(m)}(t,\varepsilon)$ and the subsequent evaluation of the resulting expression)

Let us consider the difference

$$X(t,\varepsilon) = Z_\alpha^{(m)}(t,\varepsilon) - Z_\alpha(t,\varepsilon), \tag{38.4}$$

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* The restriction of the real vector function $Y(t)$ will be interpreted, as usual, to mean the restriction of the function

$$\|Y(t)\| = \sqrt{\sum_{j=1}^{\infty} y_j^2(t)}.$$

where $Z_\alpha^{(m)}(t,\varepsilon)$ and $Z_\alpha(t,\varepsilon)$ are the approximate and exact solutions of system (37.2), corresponding to the same initial conditions.

Obviously, the vector function $X(t,\varepsilon)$ satisfies the equation

$$\frac{d^2X}{dt^2} + \Omega X = \varepsilon A(\tau) X + \varepsilon^m R(\tau,\varepsilon) \tag{38.5}$$

and the initial condition

$$X(0,\varepsilon) = 0; \qquad \left.\frac{dX}{dt}\right|_{t=0} = 0. \tag{38.6}$$

Theorem 6.2. Let us assume that the conditions of Theorem 6.1 are satisfied. We also assume that the exact solution $Z_\alpha(t,\varepsilon)$ and the m-th approximation to this solution--$Z_\alpha^{(m)}(t,\varepsilon)$--are taken for the same initial conditions. Then we can find such positive numbers C and ε_1 ($\varepsilon_1 \leq \varepsilon_0$) so that when $\varepsilon < \varepsilon_1$ the following inequality will be satisfied in the interval $0 \leq t \leq \frac{L}{\varepsilon}$:

$$\| Z_\alpha^{(m)}(t,\varepsilon) - Z_\alpha(t,\varepsilon) \| \leqslant C\varepsilon^{m-1}. \tag{38.7}$$

Proof. Let us represent problem (38.5) and (38.6) in the coordinate form

$$\frac{d^2x_n}{dt^2} + \omega_n^2 x_n = \varepsilon \sum_{j=1}^{\infty} A_{nj}(\tau) x_j + \varepsilon^m r_n(\tau,\varepsilon), \tag{38.8}$$

$$x_n(0,\varepsilon) = 0, \qquad \left.\frac{dx_n}{dt}\right|_{t=0} = 0. \tag{38.9}$$

It is easy to see that the system of differential equations (38.8) with initial conditions (38.9) is equivalent to the following system of integral equations /244

$$x_n(t) = \frac{\varepsilon}{\omega_n}\int_0^t \sin\omega_n(t-s)\left[\sum_{j=1}^{\infty} A_{nj}(\sigma) x_j(s,\varepsilon)\right] ds + \tag{38.10}$$

$$+ \frac{\varepsilon^m}{\omega_n}\int_0^t \sin\omega_n(t-s) r_n(\sigma,\varepsilon)\, ds \qquad (\sigma = \varepsilon s),\ n = 1, 2, \ldots.$$

Using the Cauchy-Bunyakovskiy inequality and its analog for definite integrals, as a result of (38.10), we shall obtain the estimate

$$x_n^2(t,\varepsilon) \leqslant 2\left\{\frac{\varepsilon L}{\omega_n^2}\int_0^t\left[\sum_{j=1}^{\infty}A_{nj}(\sigma)\sum_{i=1}^{\infty}x_i^2(s,\varepsilon)\right]ds+\frac{\varepsilon^{2m-1}L}{\omega_n^2}\int_0^t r_n^2(\sigma,\varepsilon)ds\right\}. \qquad (38.11)$$

In view of the uniform boundedness of the expression

$$\left|\sum_{j=1}^{\infty}A_{nj}^2(\tau)\right| \leqslant K = \text{const}$$

in [0,L], (see (36.12)) and the convergence of series $\sum_{n=1}^{\infty}\frac{1}{\omega_n^2} = M$, the following inequality follows from estimate (38.11):

$$\|X(t,\varepsilon)\|^2 \leqslant 2\varepsilon LMK\int_0^t\|X(s,\varepsilon)\|^2 ds + 2\varepsilon^{2m-1}LM\int_0^t\|R(\sigma,\varepsilon)\|^2 ds. \qquad (38.12)$$

Now, by utilizing the results of Lemma 1.1, we obtain

$$\|X(t,\varepsilon)\|^2 \leqslant 2\varepsilon^{2m-1}LM\int_0^t e^{2\varepsilon LMK(t-s)}\|R(\sigma,\varepsilon)\|^2 ds$$

or

$$\|X(t,\varepsilon)\| \leqslant C\varepsilon^{m-1}, \qquad (38.13)$$

where

$$C = \sqrt{2e^{2L^2MK}L^2MR^*}, \qquad R^* = \max_{0\leqslant\tau\leqslant L}[\|R(\tau,\varepsilon)\|^2].$$

According to (38.4), inequality (38.13) is the unknown inequality (38.7). Theorem 6.2 is proven, i.e., we have shown the asymptotic convergence of the approximation solution to the exact solution.

When "resonance" is absent, estimate (38.7) improves for the particular solution of the homogeneous system and also when the perturbation force is small: $\varepsilon p(x,\tau)e^{i\theta(t,\varepsilon)}$ (see equation (36.1)). This is so because the m-th approximation in the specified cases satisfies the initial system with an accuracy determined by a quantity of order $O(\varepsilon^{m+1})$ and not of order $O(\varepsilon^m)$ as determined by us when conditions of Lemma 6.1 were satisfied. /245

In conclusion we note that in practice, when we solve a specific problem

the following approach is frequently used. The infinite system (37.1) is "partitioned" at some number N, and we limit ourselves to the investigation of solutions for a finite system consisting of N differential equations of the second order. The asymptotic solutions of this "shortened" system are constructed using the method described in the present chapter. A comparison of results obtained in this manner with results obtained by other methods indicates good agreement.

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